PROJECTOR DECOMPOSITION OF R-MATRICES AND q-DEFORMED MINKOWSKI SPACE[1]

M. WOHLGENANNT

Institut für heoretische Physik Ludwig-Maximilians-Universität München
(Theresienstrasse 37, 80333 München, Germany);
e-mail: Michael.Wohlgenannt@Physik.Uni-Muenchen.de)

UDC 538.9; 538.915; 517.957

The projector decompostion of R-matrices is discussed in general and used to construct quantum spaces. This method of constructing quantum spaces is applied to the case of the q-deformed Lorentz group.

1. Introduction

Non-commutative structures have attracted more and more interest in the last few years [1, 2]. In this talk, I want to introduce quantum groups as q-deformations of the function algebra over classical Lie groups and comodules to such quantum groups. The process of q-deformation is well established [4]. As examples, we want to consider the quantum groups $\mathrm{SL}_q(N)$, $\mathrm{SO}_q(N)$, and the quantum Lorentz group. The aim is to formulate coordinate spaces whose underlying symmetry structure is not a Lie group any more, but a quantum group. Therefore, these coordinate spaces are identified with comodule algebras with respect to the relevant quantum groups. In this way, the notion of coordinate spaces is generalized. In the limit $q \to 1$, the usual commutative coordinate spaces are regained. In general, Hilbert space representations of the coordinate space algebra will have a discrete spectrum. This approach can therefore certainly lead to a regularization scheme for gauge theories. Gauge theories on non-commutative spaces will resemble a gauge theory on some lattice. A lot of ongoing work is concerned with formulating gauge field theories on non-commutative spaces.

2. Quantum Groups

Let us consider the function algebra over some classical matrix group of $n \times n$ square matrices, which is most interesting from a physical point of view [3, 4]. The algebra is generated by the coordinate functions

$$t^i_j, \qquad i,j = 1,\dots,n. \tag{1}$$

The q-deformation of this commutative function algebra $\mathcal{A}_q$ is defined as

$$\mathcal{A}_q := \frac{\mathbb{C} < t^i_j >}{I_R}, \tag{2}$$

where I_R is a 2-sided ideal generated by the (RTT) relations

$$R^{ij}_{mp} t^m_k t^p_l = t^j_p t^i_m R^{mp}_{kl}, \tag{3}$$

where R is a $n^2 \times n^2$ matrix (depending on the real deformation parameter q) satisfying the Quantum-Yang-Baxter-Equation (QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{4}$$

where, e.g., $R_{13}{}^{ijk}_{lmn} = \delta^j_m R^{ik}_{ln}$.

2.1. Quantum Planes

Let $f(\hat{R})$ be a polynomial in $\hat{R} = PR$, $Pu \otimes v = v \otimes u$ (permutation matrix), $v, u \in \mathbb{C}^n$. Denote by I_f the 2-sided ideal in $\mathbb{C} < x^1, \dots, x^n >$ generated by the relations

$$f(\hat{R})^{ij}_{mn} x^m \otimes x^n = 0, \quad i,j = 1,\dots,n, \tag{5}$$

summing over repeated indices. The quotient algebra $H = \frac{\mathbb{C}<x^1,\dots,x^n>}{I_f}$ is called **quantum plane**, due to the left co-action

$$\begin{aligned} \delta : H &\longrightarrow \mathcal{A}_q \otimes H, \\ \delta(x^i) &= t^i_j \otimes x^j. \end{aligned}$$

The RTT relations (3) imply

$$f(\hat{R})^{ij}_{mp} t^m_k t^p_l = t^i_m t^j_p f(\hat{R})^{mp}_{kl}. \tag{6}$$

[1] Presented at the XIIIth International Hutsulian Workshop "Methods of Theoretical and Mathematical Physics" (September 11 — 24 2000, Uzhgorod — Kyiv — Ivano-Frankivsk — Rakhiv, Ukraine) and dedicated to Prof. Dr. W. Kummer on the occasion of his 65th birthday.

Define an algebraic structure on H via

$$f(\hat{R})^{ij}_{mn} x^m \otimes x^n = 0. \tag{7}$$

The co-action of $\mathcal{A}_q$ is compatible with this structure on H, since we have

$$f(\hat{R})^{ij}_{mn} \delta(x)^m \otimes \delta(x)^n = 0. \tag{8}$$

The matrix $\hat{R}$ is the q-deformation of the permutation matrix. SO we look at the spectral decomposition of $\hat{R}$, which will consist of quantum analogues of the symmetrizer and antisymmetrizer. In analogy to the commutative coordinates, we choose $f(\hat{R}) = P_A$, where P_A is the "antisymmetrizer". In the limit $q \to 1$, we have ordinary commuting coordinates.

$$P_A{}^{ij}_{kl} x^k \otimes x^l = 0. \tag{9}$$

3. $\mathbf{SL}_q(N)$

Let us consider the q-deformation of the function algebra over the Lie group $\mathrm{SL}(N, \mathbb{C})$ as a first example. Let

$$\begin{array}{ccc} \mathcal{R} : \mathrm{SL}_q(N) & \otimes & \mathrm{SL}_q(N) \\ & \downarrow & \\ & \mathbb{C} & \end{array}$$

be the dual quasitriangular structure of $\mathrm{SL}_q(N)$. Then the R-matrix is defined by

$$R^{ij}_{kl} := \mathcal{R}(t^i_j \otimes t^j_l), \tag{10}$$

where t^i_j is a coordinate function [3]. Again we have $\hat{R} := PR$. The $N^2 \times N^2$ matrix $\hat{R}$ has - for the defining representation - the spectral decomposition

$$\hat{R} = qP_S - q^{-1} P_A,$$

where

$$\begin{aligned} P_S &= \frac{\hat{R} + q^{-1}\mathbf{1}}{q + q^{-1}} \text{ — symmetric projector,} \\ P_A &= \frac{-\hat{R} + q\mathbf{1}}{q + q^{-1}} \text{ — antisymmetric projector.} \end{aligned}$$

3.1. Comodule Space

The coordinate space, whose underlying symmetry structure is given by the quantum group $\mathrm{SL}_q(N)$, is defined by relation (9)

$$P_A x \otimes x = 0, \tag{11}$$

which generates the ideal I_{P_A}. The comodule algebra is given by

$$H = \frac{\mathbb{C} < x^1, \ldots, x^n >}{I_{P_A}}, \tag{12}$$

Explicitly, the generators x^i fulfill the relations

$$x^i x^j = q x^j x^i, \quad 1 \le i < j \le N. \tag{13}$$

$\mathrm{SL}_q(N)$ is a Hopf algebra with the following linear structure maps which generalize matrix notation,

$$\Delta t^i_j = t^i_k \otimes t^k_j, \tag{14}$$

$$\epsilon(t^i_j) = \delta^i_j, \tag{15}$$

$$S(T) = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}, \tag{16}$$

$$T = \left(t^i_j\right)^N_{i,j=1}, \tag{17}$$

with the comultiplication $\Delta : \mathrm{SL}_q(N) \longrightarrow \mathrm{SL}_q(N) \otimes \mathrm{SL}_q(N)$, the counit $\epsilon : \mathrm{SL}_q(N) \longrightarrow \mathbb{C}$ and the antipode $S : \mathrm{SL}_q(N) \longrightarrow \mathrm{SL}_q(N)$. q is the deformation parameter. And we have the following limit

$$\mathrm{SL}_q(N) \longrightarrow \mathrm{SL}(N, \mathbb{C}), \text{ for } q \to 1.$$

4. $\mathbf{SO}_q(N)$

The $\hat{R}$ matrix of the quantum group $\mathrm{SO}_q(N)$ has the following spectral resolution, for $N > 2$ and $(1+q^2)(1+q^{2-n})(1-q^{-N}) \neq 0$,

$$\hat{R} = qP_S - q^{-1}P_A + q^{1-N}P_0, \tag{18}$$

where

$$P_S = \frac{\hat{R}^2 - (q^{1-N} - q^{-1})\hat{R} - q^{-N}\mathbf{1}}{(q + q^{-1})(q - q^{1-N})}, \tag{19}$$

$$P_A = \frac{\hat{R}^2 - (q^{1-N} + q)\hat{R} + q^{-N+2}\mathbf{1}}{(q + q^{-1})(q - q^{1-N})}, \tag{20}$$

$$P_0 = \frac{\hat{R}^2 - (q - q^{-1})\hat{R} - \mathbf{1}}{(q^{1-N} - q)(q^{-1} + q^{1-N})}. \tag{21}$$

Again $\mathrm{SO}_q(N)$ is a Hopf algebra with the antipode map

$$S(T) = CT^T C^{-1}, \tag{22}$$

where C is the metric of $\mathrm{SO}_q(N)$.

4.1. Comodule Space

Again the coordinate algebra, whose underlying symmetry structure is $\mathrm{SO}_q(N)$, is defined by the relations

$$P_A x \otimes x = 0. \tag{23}$$

The coordinate algebra is given by

$$H = \frac{\mathbb{C} < x^1, \ldots, x^n >}{I_{P_A}}. \tag{24}$$

Eqn. (23) reads in explicit form

$$x^i x^j = q x^j x^i, \qquad 1 \le i < j \le N, i \ne j' \tag{25}$$

$$x^{i'} x^i = x^i x^{i'} + (q^2-1) \sum_{j=1}^{i'-1} q^{\rho_{i'} - \rho_j} x^j x^{j'} - \tag{26}$$

$$- \frac{q^2-1}{1+q^{N-2}}\, q^{\rho_{i'}} X^T C X, \tag{27}$$

where $i' = N+1-i$, $j' = N+1-j$ and

$$(\rho_1, \ldots, \rho_N) = \begin{cases} (n-\frac{1}{2}, n-\frac{3}{2}, \ldots, -n+\frac{1}{2}), & \text{for } N = 2n+1 \\ (n-1, n-2, \ldots, 1, 0, 0, -1, \ldots, -n+1), & \text{for } N = 2n. \end{cases}$$

5. q-Deformed Lorentz Group

We define the q-deformed Lorentz group as the tensor product of 2 different copies of $\mathrm{SL}_q(2)$ quantum groups [5]

$$\mathcal{A}_q := \widetilde{\mathrm{SL}_q(2)} \otimes \mathrm{SL}_q(2). \tag{28}$$

In order to differ between the two identical copies, we have introduced the "tilde" above the former. The construction in [5] follows the classical construction of 4-vectors using 2-spinors. $\widetilde{\mathrm{SL}_q(2)}$ has to be identified as the Hermitian conjugate of $\mathrm{SL}_q(2)$. We have the following relations:

$$\hat{R}^{ij}_{kl}\, t^k_r\, t^l_s = t^i_k\, t^j_l\, \hat{R}^{kl}_{rs}, \tag{29}$$

$$\hat{R}^{ij}_{kl}\, \tilde{t}^k_r\, \tilde{t}^l_s = \tilde{t}^i_k\, \tilde{t}^j_l\, \hat{R}^{kl}_{rs}, \tag{30}$$

which are just saying that $\mathrm{SL}_q(2)$ and $\widetilde{\mathrm{SL}_q(2)}$ are quantum groups with "braiding" $\hat{R}$, generated by the coordinate functions t^i_j and $\tilde{t}^i_j$, respectively. Further one needs to know consistent relations between the two algebras. They are defined by [5]

$$\hat{R}^{ij}_{kl}\, \tilde{t}^k_r\, t^l_s = t^i_k\, \tilde{t}^j_l\, \hat{R}^{kl}_{rs}. \tag{31}$$

5.1. Comodule Space

The generators of q-deformed Minkowski space, i.e., the $\mathcal{A}_q$-comodule algebra (cf. (28)), are given by

$$X^{ij} := \tilde{x}^i x^j, \tag{32}$$

with the left co-action

$$\delta(X^{ij}) = \tilde{t}^i_k\, t^j_l \otimes X^{kl}. \tag{33}$$

Let us define the generators of the q-deformed Lorentz group

$$\Lambda^{ij}_{km} := \tilde{t}^i_k\, t^j_m. \tag{34}$$

We want to compute the algebra relations in view of (32). To this aim, we need to calculate the $\hat{R}_L$-matrix for the q-deformed Lorentz group and decompose it into projectors. From (29—31) and the requirement

$$\Lambda^{ij}_{kl}\, \Lambda^{mn}_{op}\, \hat{R}_L^{(kl)(op)}{}_{(rs)(tu)} = \hat{R}_L^{(ij)(mn)}{}_{(kl)(op)}\, \Lambda^{kl}_{rs}\, \Lambda^{op}_{tu}, \tag{35}$$

we get the $\hat{R}_L$-matrix of the q-deformed Lorentz group ...

$$\hat{R}_L^{(ij)(kl)}{}_{(k'l')(i'j')} = \frac{1}{q} \hat{R}^{jk}_{k''j''}\, \hat{R}^{ik''}_{k'i''}\, \hat{R}^{j''l}_{l''j'}\, \hat{R}^{-1}{}^{i''l''}_{l'i'} \tag{36}$$

$\hat{R}_L$ has the same spectral decomposition as the $\hat{R}$-matrix of $\mathrm{SO}_q(4)$:

$$(\hat{R}_L - q\mathbf{1})(\hat{R}_L + \frac{1}{q}\mathbf{1})(\hat{R}_L - \frac{1}{q^{-3}}\mathbf{1}) = 0, \tag{37}$$

$$\hat{R}_L = q P_S - \frac{1}{q} P_A + \frac{1}{q^{-3}} P_0, \tag{38}$$

$$\left.\begin{array}{l} P_S = \\ P_A = \\ P_0 = \end{array}\right. \text{dependence on } \hat{R}_L \text{ as in the } \mathrm{SO}q(4) \text{ case} \tag{39}$$

However, P_S, P_A, P_0, and therefore $\hat{R}_L$ are numerically different.

5.2. q-Deformed Minkowski Plane

Now that we have the spectral decomposition of $\hat{R}_L$, we are able to calculate the expression

$$P_A X \otimes X = 0. \tag{40}$$

First of all, let us use the light-cone coordinates $(A, B, C, D) = (X_1^2, X_2^1, X_1^1, X_2^2)$. In this basis, relations (40) read

$$\begin{array}{l} AB = BA + (q^{-2} - 1)CD + (q^2 - 1)DD, \\ AC = CA + (q^2 - 1)AD, \\ BC = CB + (1 - q^{-2})DB, \\ BD = q^2 DB, \\ DA = q^2 AD, \\ CD = DC. \end{array} \tag{41}$$

Redefinition of the coordinates

$$X^0 := \frac{C + D}{q\sqrt{q + \frac{1}{q}}},$$

$$X^+ := \frac{A}{\sqrt{q}},$$

$$X^- := \sqrt{q}B,$$

$$X^3 := \frac{\frac{1}{q}C - qD}{\sqrt{q + \frac{1}{q}}} \tag{42}$$

leads to the following relations:

$$X^0 X^3 = X^3 X^0,$$

$$X^0 X^+ = X^+ X^0,$$

$$X^0 X^- = X^- X^0,$$

$$X^3 X^+ - q^2 X^+ X^3 = (1 - q^2) X^+ X^0,$$

$$X^- X^3 - q^2 X^3 X^- = (1 - q^2) X^0 X^-,$$

$$X^+ X^- - X^- X^+ = (q - \frac{1}{q})(X^3 X^3 - X^3 X^0). \tag{43}$$

However, one can construct another $\hat{R}$-matrix for the q-deformed Lorentz group, ${}_{II}\hat{R}_L$. The second matrix ${}_{II}\hat{R}_L$ has been constructed in [6], where it was found by decomposing $\hat{R}_L$ into projectors and taking a different combination of them. In terms of the $\mathrm{SL}_q(2)$-$\hat{R}$-matrix, we have [7]

$${}_{II}\hat{R}_L^{(ij)(kl)}{}_{(k'l')(i'j')} = \hat{R}^{-1}{}^{li}_{ba}\hat{R}^{-1}{}^{kb}_{k'c}\hat{R}^{aj}_{dj'}\hat{R}^{cd}_{l'i'}. \tag{44}$$

Both $\hat{R}$-matrices are needed to introduce derivatives (i.e., momenta) [6]. In terms of projectors, the two $\hat{R}$ matrices read

$${}_{II}\hat{R}_L = qP_S + qP_0 - q^3 P_- - q^{-1} P_+,$$

$$\hat{R}_L = qP_S + q^{-3} P_0 - q^{-1} P_- - q^{-1} P_+,$$

where $P_A \equiv P_+ + P_-$.

6. Summary and Outlook

We have introduced quantum groups as q-deformation of the function algebra over Lie groups and coordinate algebras whose internal symmetry is given by quantum groups. The most important example we have considered is q-deformed Lorentz group and q-deformed Minkowski space. So far these facts are not new. The intension of this talk is to introduce this method of constructing quantum spaces and to introduce q-deformed Minkowski space. In this respect, this talk should give a better understanding of the talk of my colleague Fabian Bachmaier, *q-Deformed Minkowski Space*. He will talk about representations of the Lorentz algebra and the Minkowski algebra (41). Ongoing research is concerned with constructing a field theory on quantum spaces, cf. [8 — 10] and references therein.

1. *Madore J., Schraml S., Schupp P., Wess J.*// Eur.Phys.J.– 2000. – **C16**. – P. 161–167; hep-th/0001203.
2. *Seiberg N., Witten E.*// JHEP. – 1999. – **9909**. –P. 032, hep-th/9908142.
3. *Majid S.* Foundations of Quantum Group Theory. – Cambridge University Press, 1995.
4. *Reshetikhin N.Yu., Takhtadzhyan L.A., Faddeev L.D.*// Leningrad Math. J. – 1990. – **1**. – N 1.
5. *Carow-Watamura U., Schlieker M., Scholl M., Watamura S.*// Intern. J. Mod. Phys. A. – 1991. – **6**. – P. 3081.
6. *Lorek A., Schmidke W.B., Wess J.*// Lett. Math. Phys. – 1994. – **31**. – P. 279.
7. *Majid S., Meyer U.*// Z. Phys. C. – 1994. – **63**. – P. 357; hep-th/9312170.
8. *Jurco B., Schupp P., Wess J.*// Nucl. Phys. B. – 2000. – **584**. – P. 784–794; hep-th/0005005.

9. *Jurco B., Schraml S., Schupp P., Wess J.*// Eur. Phys. J. C. – 2000. – **17**. – P. 521–526; hep-th/0006246.

10. *Wachter H., Wohlgenannt M.* *-Products on Quantum Spaces, hep-th/0103120.

ПРЕКЦІЙНЕ РОЗКЛАДАННЯ R-МАТРИЦЬ
І q-ДЕФОРМОВАНИЙ ПРОСТІР
МІНКОВСЬКОГО

М. Волгенант

Р е з ю м е

Розглянуто загальний випадок проекційного розкладання R-матриць та його використання для побудови квантових просторів. Для q-деформованої групи Лоренца запропонований метод застосовано для побудови просторів в явному вигляді.

ПРОЕКЦИОННОЕ РАЗЛОЖЕНИЕ R-МАТРИЦ
И q-ДЕФОРМИРОВАННОЕ ПРОСТРАНСТВО
МИНКОВСКОГО

М. Волгенант

Р е з ю м е

Рассматривается общий случай проенкционного разложения R-матриц и его использование для построения квантовых пространств. Для q-деформированной группы Лоренца предложенный метод применен для построения квантовых пространств в явном виде.