GENERALIZED CLEBSH AND NEUMAN INTEGRABLE SYSTEMS FROM THE SPECIAL QUASIGRADED LIE ALGEBRAS ON HIGHER GENUS CURVES[1]

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In the present paper, we construct special quasigraded so(n)-valued Lie algebras on higher genus algebraic curves. Using them, we find a new Lie theoretical interpretation and new Lax type representations with spectral parameter for the generalized Clebsh and Neuman systems.

1. Introduction

Graded Lie algebras play important role in the theory of integrable Hamiltonian systems [1, 2]. This is explained by the fact that loop algebras admit the so-called Kostant—Adler scheme [3], that is, by far the mightiest method of producing a wide set of mutually comuting integrals of motion.

In [4, 5], new examples of infinite-dimensional Lie algebras that admit Kostant—Adler scheme were constructed. They coincide with the special so(3) valued algebra of meromorphic functions on an elliptic curve.

In our previous papers [7, 8] we gave the higher rank generalizations of the algebraic construction of [5]. In more detail, we have constructed $\mathbb{Z}$ quasigraded infinite dimensional Lie algebras $\widetilde{\mathfrak{g}}_{H_n}$ associated with higher genus curves H_n and classical matrix Lie algebra $\mathfrak{g}$. Using them, we have produced new integrable finite-dimensional systems, generalizing Steklov—Veselov and Steklov—Lyapunov integrable systems.

In the present paper, we consider the special degeneration H'_n of the curve H_n, corresponding Lie algebras $\widetilde{\mathrm{so}(n)}_{H'_n}$, and integrable finite-dimensional Hamiltonian systems associated with them. It occurred that such integrable systems as generalized Clebsh and Neuman systems could be naturally obtained from these algebras. It is necessary to mention that generalized Clebsh and Neuman systems could be also obtained using the approach of [1, 2] based on loop algebras, but, in this case, in order to obtain these systems, an additional Hamiltonian reduction [2] is needed. On the contrary, in case of our algebras, no additional Hamiltonian reduction is needed for obtaining Clebsh and Neuman systems. That is why, we consider our algebras to be more naturally connected with these systems.

The structure of the present article is the following: in Section 2, we describe the Lie algebras $\widetilde{\mathrm{so}(n)}_{H_n}$ and $\widetilde{\mathrm{so}(n)}_{H'_n}$, their dual spaces and invariants of coadjoint representations. In Section 3, a general framework of obtaining integrable Hamiltonian systems using the above algebras is exposed, and some properties of these systems are investigated. In the last section, the examples of generalized Clebsh and Neuman systems are considered.

2. Special Quasigraded Lie Algebras

A hyperelliptic curve embedded in $\mathbb{C}^n$. Let us consider the following system of quadrics in the space $\mathbb{C}^n$ with the coordinates $w_1, w_2, ..., w_n$:

$$w_i^2 - w_j^2 = a_j - a_i, \quad i,j = 1,n, \tag{1}$$

where a_i are arbitrary complex numbers. The rank of this system is $n-1$, so the substitution

$$w_i^2 = w - a_i, \quad y = \prod_{i=1}^{n} w_i$$

solves these equations and defines the equation of a hyperelliptic curve $\mathcal{H}$. Hence, Eqs. (1) define embedding a hyperelliptic curve $\mathcal{H}$ in the linear space $\mathbb{C}^n$.

Example 1: In the $n = 3$ case, all of these objects have the well-known analytic description. Indeed, in this case, the curve under consideration is elliptic. Its uniformization is made by the Weierstrass $\mathfrak{p}$ -function

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and its derivative: $w = \mathfrak{p}(u)$, $y = 1/2\mathfrak{p}'(u)$. Functions w_i are expressed via Jacobi elliptic functions.

Canonical basis of $\mathrm{so}(n)$**.** Let $\mathfrak{g}$ denote the algebra $\mathrm{so}(n)$ over the field $\mathbb{R}$ or $\mathbb{C}$. Let $I_{i,j} \in \mathrm{Mat}(n,\mathbb{R})$ be a matrix defined as follows:

$$(I_{ij})_{ab} = \delta_{ia}\delta_{jb}.$$

Evidently, a basis in the algebra $\mathrm{so}(n)$ could be chosen as: $X_{ij} \equiv I_{ij} - I_{i,j}$, $i,j \in 1,...,n$, with the commutation relations

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j} + \delta_{j,l}X_{k,i} - \delta_{k,i}X_{j,l},$$

and "skew-symmetry" property $X_{ij} = -X_{ji}$.

Lie algebras on the curve $\mathcal{H}_n$**.** For the basic elements X_{ij} of the algebra $\mathrm{so}(n)$, we introduce the following algebra-valued functions on the curve $\mathcal{H}_n$ or, to be more precise, on its double covering:

$$X_{ij}^m(w) = X_{ij} \otimes w^m w_i w_j, \quad i,j \in \overline{1,n};\ m \in \mathbb{Z} \tag{2}$$

The next theorem holds true:

Theorem 1. *(i) Elements* X_{ij}^r , $r \in \mathbb{Z}$ *form* $\mathbb{Z}$ *quasi-graded Lie algebra* $\widetilde{\mathrm{so}(n)}_{\mathcal{H}_n}$ *with the following commutation relations:*

$$[X_{ij}^r, X_{kl}^s] = \delta_{kj}X_{il}^{r+s+1} - \delta_{il}X_{kj}^{r+s+1} + \delta_{jl}X_{ki}^{r+s+1} -$$

$$-\delta_{ik}X_{jl}^{r+s+1} + a_i\delta_{il}X_{kj}^{r+s} - a_j\delta_{kj}X_{il}^{r+s} + a_i\delta_{ik}X_{jl}^{r+s} -$$

$$-a_j\delta_{jl}X_{ki}^{r+s}. \tag{3}$$

(ii) Algebra $\widetilde{\mathrm{so}(n)}_{\mathcal{H}_n}$ *as a linear space admits a decomposition into the direct sum of two subalgebras:* $\widetilde{\mathrm{so}(n)}_{\mathcal{H}_n} = \widetilde{\mathrm{so}(n)}^{+}_{\mathcal{H}_n} + \widetilde{\mathrm{so}(n)}^{-}_{\mathcal{H}_n}$, *where the subalgebras* $\widetilde{\mathrm{so}(n)}^{+}_{\mathcal{H}_n}$ *and* $\widetilde{\mathrm{so}(n)}^{-}_{\mathcal{H}_n}$ *are generated by the elements* X_{ij}^0, *and* X_{ij}^{-1}, *correspondingly.*

(See [8] for the proof of Theorem 1).

The algebras constructed in Theorem 1 depend on n complex numbers a_i — branching points of the curve $\mathcal{H}_n$. We may impose different constraints on the numbers a_i, i.e., consider different degenerations on the curve $\mathcal{H}_n$ in order to obtain different algebraic structures that will lead to different integrable systems as a result.

Special degeneration of the curve $\mathcal{H}_n$ **and "anisotropic affine algebra".** Let us consider special case of the algebra $\widetilde{\mathrm{so}(n)}_{\mathcal{H}_n}$ in the case where one of the branching points a_i turns to zero. Let us put, for definiteness, $a_n = 0$, $a_i \neq 0$ for $i = 1, n-1$. We denote such curves $\mathcal{H}'_n$. Let us introduce the following notations:

$$X_{ij}^m = X_{ij} \otimes w^m w_i w_j,\ Y_i^m = X_{in} \otimes w^{m+1/2} w_i, \tag{4}$$

where $i,j \in \overline{1,n-1}$; $m \in \mathbb{Z}$.

The following corollary of Theorem 1 holds:

Corollary 1. *Generators* Y_i^r, X_{ij}^s *where* $i,j,k,l \in \overline{1,n-1}$, $s,r \in \mathbb{Z}$, *satisfy the following commutation relations:*

$$[X_{ij}^s, X_{kl}^r] = \delta_{kj}X_{il}^{s+r+1} - \delta_{il}X_{kj}^{s+r+1} + \delta_{jl}X_{ki}^{s+r+1} -$$

$$-\delta_{ik}X_{jl}^{s+r+1} + a_i\delta_{il}X_{kj}^{s+r} - a_j\delta_{kj}X_{il}^{s+r} + a_i\delta_{ik}X_{jl}^{s+r} -$$

$$-a_j\delta_{jl}X_{ki}^{s+r}, \tag{5a}$$

$$[X_{ij}^s, Y_k^r] = \delta_{kj}Y_i^{s+r+1} - \delta_{ik}Y_j^{s+r+1} - a_j\delta_{kj}Y_i^{s+r} +$$

$$+a_i\delta_{ik}Y_j^{s+r}, \tag{5b}$$

$$[Y_i^s, Y_k^r] = X_{ki}^{s+r+1}, \tag{5c}$$

and form a basis in the $\mathbb{Z}$ *quasi-graded,* Z_2 *graded Lie algebra* $\widetilde{\mathrm{so}(n)}_{\mathcal{H}'_n}$.

Remark 1. Special attention devoted to this algebra is explained by the fact that it will produce a hyperelliptic Lax pair for the generalized Clebsh and Neuman systems. As will be shown below, it coincides with the higher-rank generalization of the so(3) "anisotropic affine algebra" introduced in [4] in connection with the Landau—Lifshitz equations.

Example 2. Let us consider the $n = 4$ example. In this case, the commutation relations (5) could be much more simplified by introducing a standard basis in the $\mathrm{so}(3) \subset \mathrm{so}(4)$ algebra. Putting $X_k \equiv \epsilon_{ijk}X_{ij}$, we obtain the following commutation relations for $\widetilde{\mathrm{so}(4)}_{\mathcal{H}'_4}$:

$$[X_i^r, X_j^s] = \epsilon_{ijk}X_k^{r+s+1} + \epsilon_{ijk}a_kX_k^{r+s}, \tag{6a}$$

$$[X_i^r, Y_j^r] = \epsilon_{ijk}Y_k^{r+s+1} + \epsilon_{ijk}a_jY_k^{r+s}, \tag{6b}$$

$$[Y_i^r, Y_j^s] = \epsilon_{ijk} X_k^{r+s+1}. \tag{6c}$$

Introducing new notations $A_i^{2s} \equiv X_i^{s-1}$, $A_j^{2r-1} \equiv Y_j^{r-1}$, we see that, up to a minor change of quasi-gradation, the algebra $\widetilde{\mathrm{so}(4)}_{\mathcal{H}_4'}$ coincides with the "anisotropic affine algebra" in the form of [4].

Remark 2. It is, of course, possible to consider other degenerations of the curve $\mathcal{H}_n$ and consider correspondent algebras. In particular, when all numbers $a_i \to 0$, i.e., in the rational degeneration, we obtain loop algebras. Their usage in the theory of integrable systems was extensively studied in [1, 2].

Coadjoint representation and its invariants. In order to define Hamiltonian systems, we have to define the coadjoint representation and space $\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n}$. We assume, that $\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n} \subset \mathrm{so}(n) \otimes A$, where A is an algebra of functions on the double covering of the curve $\mathcal{H}_n$.

Let (,) denotes the standard Killing—Kartan form on so(n) — for the elements $X = \sum\limits_{i,j=1}^{n} x_{ij} X_{ij}$, $Y = \sum\limits_{i,j=1}^{n} y_{ij} X_{ij}$, we have $(X,Y) = 1/2Tr(XY) = \sum\limits_{i,j=1}^{n} x_{ij} y_{ij}$. Let us define pairings between $L(w) \in \widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n}$ and $X(w) \in \widetilde{\mathrm{so}(n)}_{\mathcal{H}_n}$ as follows:

$$\langle X(w), L(w) \rangle_K = res_{w=0} w^{-K} y^{-1}(w) (X(w), L(w)). \tag{7}$$

We denote these pairings by $\langle\ ,\ \rangle_K$. Under all of the pairings described above, the Lax operator will have the following form:

$$L(w) = \sum_{m\in Z} \sum_{i,j=1}^{d} l_{ij}^{(m)} w^{m-1} \frac{y(w)}{w_i w_j} X^*_{ij}. \tag{8}$$

where the coefficient functions $l_{ij}^{(k)}$ satisfy the skew-symmetry conditions $l_{ij}^{(k)} = -l_{ji}^{(k)}$.

It follows from the explicit form of the pairing that the action of the algebra $\widetilde{\mathrm{so}(n)}_{\mathcal{H}_n}$ on its dual space $\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n}$ coincides with the commutator:

$$ad^*_{X(w)} L(w) = [L(w), X(w)], \tag{9}$$

that, in turn, entails the next statement:

Proposition 1. *Functions $H^{2k}(L(w)) = \mathrm{tr} L(w)^{2k}$, where $k \in \overline{0, [n/2]}$, are generating functions of the invariants of the coadjoint representation.*

Conclusion. The constructed Lie algebras admit decomposition into the direct sum of two subalgebras and possess the infinite number of invariant functions. These two properties permit their usage in the theory of integrable Hamiltonian systems. We will demonstrate this in the next section.

3. Hamiltonian Systems via Quasi-Graded Algebras

In this section, we will construct Hamiltonian systems that correspond to the algebras $\widetilde{\mathrm{so}(n)}_{\mathcal{H}_n}$ and $\widetilde{\mathrm{so}(n)}_{\mathcal{H}_n'}$ To do this, we define Lie—Poisson structures and Lie—Poisson subspaces. All formulas will be explicitly written in the case of the algebra $\widetilde{\mathrm{so}(n)}_{\mathcal{H}_n}$. The corresponding formulas for $\widetilde{\mathrm{so}(n)}_{\mathcal{H}_n'}$ could be obtained taking the continuous limit $a_n \to 0$.

Lie—Poisson structures. In the space $\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n}$, one can define many Lie—Poisson structures using pairings $\langle\ ,\ \rangle_K$ for different K. They define brackets on $P(\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n})$ (and $P(\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n'})$) in the following way:

$$\{F(L(\lambda)), G(L(\lambda))\}_K = \langle L(\lambda), [\nabla_K F, \nabla_K G] \rangle_K, \tag{10}$$

where $\nabla_K F(L(\lambda)) = \sum\limits_{n\in Z} \sum\limits_{i,j=1}^{d} \dfrac{\partial F}{\partial l_{ij}^{(n)}} X_{ij}^{-n+K}$.

From Proposition 3., the next statement follows:

Proposition 2. *Functions $H_m^k(L(w))$ are central for brackets $\{\ ,\ \}_K$.*

Let us explicitly calculate the Poisson brackets (10). It is easy to show that, for the coordinate functions $l_{ij}^{(m)}$, these brackets have the following form:

$$\{l_{ij}^{(n)}, l_{kl}^{(m)}\}_K = \delta_{kj} l_{il}^{(n+m-K-1)} - \delta_{il} l_{kj}^{(n+m-K-1)} +$$

$$+\delta_{jl} l_{ki}^{(n+m-K-1)} - \delta_{ik} l_{jl}^{(n+m-K-1)} + a_i \delta_{il} l_{kj}^{(n+m-K)} -$$

$$-a_j \delta_{kj} l_{il}^{(n+m-K)} + +a_i \delta_{ik} l_{jl}^{(n+m-K)} - a_j \delta_{jl} l_{ki}^{(n+m-K)} \tag{11}$$

Poisson subspaces. Let us now consider the following subspace $\mathcal{M}_N \subset \widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n}$:

$$\mathcal{M}_N = \sum_{m=1}^{N+1} (\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n})_m, \text{ where } (\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n})_m = \sum_{ij} l_{ij}^{(m)} Y_{ij}^m,$$

where $Y^m_{ij} \equiv \dfrac{w^{m-1}y(w)}{w_i w_j} X_{ij}$.

The following proposition holds true:

Proposition 3. *Brackets $\{\ ,\ \}_K$, where $K = 0$ and $K = N+1$, could be correctly restricted onto the space $\mathcal{M}_N$.*

Proof. From the explicit form of the Poisson brackets (10), it follows that the subspaces $\mathcal{M}_{1,\infty} = \sum\limits_{m=1}^{\infty} (\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n})_m$ and $\mathcal{M}_{-\infty,N} = \sum\limits_{m=-\infty}^{N+1} (\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n})_m$ are Lie—Poisson subalgebras with respect to the brackets $\{\ ,\ \}_0$ and $\{\ ,\ \}_{N+1}$, correspondingly (it is not difficult to show that these Poisson algebras are isomorphic to $\widetilde{\mathrm{so}(n)}^-_{\mathcal{H}_n}$ and $\widetilde{\mathrm{so}(n)}^+_{\mathcal{H}_n}$). The explicit form of the Lie—Poisson brackets also implies that, for any integer p, subspaces $\mathcal{J}_{p,\infty} = \sum\limits_{m=p}^{\infty} (\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n})_m$ and $\mathcal{J}_{-\infty,p} = \sum\limits_{-\infty}^{p} (\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n})_m$ are ideals in the Lie—Poisson subalgebras described above. That proves Proposition 3.

The following theorem is the most important for application in the theory of integrable systems.

Theorem 2. *Let functions $\{H^k_m(L)\}$ be defined as in Proposition 3. Then they generate a commutative algebra with respect to the restriction of the brackets $\{\ ,\ \}_0$ and $\{\ ,\ \}_{N+1}$ onto $\mathcal{M}_N$.*

Proof. From the explicit form of the Poisson brackets (10), it follows that the subspaces $\mathcal{M}_{1,\infty} = \sum\limits_{m=1}^{\infty} (\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n})_m$ and $\mathcal{M}_{-\infty,N} = \sum\limits_{m=-\infty}^{N+1} (\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n})_m$ along with the complementary subspaces are also Lie—Poisson subalgebras (with respect to the correspondent brackets $\{\ ,\ \}_0$ and $\{\ ,\ \}_{N+1}$). Hence, from the Kostant—Adler scheme, it follows that the restriction of the invariant functions $H^k_m(L)$ onto the subalgebras $(\mathcal{M}_{1,\infty},\ \{\ ,\ \}_0)$ and $(\mathcal{M}_{-\infty,N},\ \{\ ,\ \}_{N+1})$ forms a commutative subalgebra with respect to the correspondent brackets. Now, to prove the theorem, it is enough to take into consideration that the subspaces $(\mathcal{M}_N,\ \{\ ,\ \}_0)$ and $(\mathcal{M}_N,\ \{\ ,\ \}_{N+1})$ coincide with the quotient algebras $\mathcal{M}_{-\infty,N}/\mathcal{J}_{-\infty,0}$ or $\mathcal{M}_{1,\infty}/\mathcal{J}_{N+2,\infty}$, and the projection on the quotient algebra is a canonical homomorphism.

That proves the theorem.

It is necessary to emphasize that, among functions $\{H^{2r}_m(L)\}$, there are "geometric invariants" — Casimir functions of the the Lie—Poisson brackets $\{\ ,\ \}_0$ or $\{\ ,\ \}_{N+1}$ and commuting integrals that generate nontrivial flows on the correspondent coadjoint orbits. The following theorem enables one to distinguish Casimir functions from the nontrivial integrals:

Theorem 3. *Let us consider functions $\{H^{2r}_s(L)\}$ restricted to the subspace $\mathcal{M}_N$. Then*
(i) functions $H^{2r}_s(L)$ are Casimir functions of the brackets $\{\ ,\ \}_0$ if

$$(2r-1)N + r(n-2) \le s \le 2rN + r(n-2),$$

(ii) functions $H^{2r}_s(L)$ are Casimir functions of the brackets $\{\ ,\ \}_{N+1}$ if

$$0 \le s \le N,$$

(iii) for the arbitrary coordinate function $l^{(n)}_{ij}$ the following identity holds true:

$$\{l^{(n)}_{ij}, H^{2r}_s\}_0 = \{H^{2r}_{s+N+1}, l^{(n)}_{ij}\}_{N+1}.$$

Proof. Let $\widetilde{X}^m_{ij}$, $\widetilde{Y}^m_{kl}$ be vector fields that correspond to the brackets $\{\ ,\ \}_0$ and $\{\ ,\ \}_{N+1}$:

$$\widetilde{X}^m_{ij} F(L(w)) \equiv \{l^{(m)}_{ij}, F(L(w))\}_0,$$

$$\widetilde{Y}^m_{ij} F(L(w)) \equiv \{l^{(m)}_{ij}, F(L(w))\}_{N+1}.$$

Proof of the theorem will be based on the following

Proposition 4. *Let $L_{ij}(w) = y(w)(w_i w_j)^{-1} \times \times \sum\limits_{m=1}^{N+1} l^{(m)}_{ij} w^{m-1}$. The following identity holds true:*

$$w_i w_j (w^l \widetilde{X}^l_{ij} + w^{l-N-1} \widetilde{Y}^l_{ij}) F =$$

$$= \sum_{m,n=1}^{d} C^{(mn)}_{(ij),(kl)} L_{mn}(w) \frac{\partial F}{\partial L_{kl}(w)},$$

where $C^{(mn)}_{(ij),(kl)}$ are the structure constants of the algebra $\mathrm{so}(d)$.

Proposition 4 is proved by direct calculations. Now, to prove Theorem 3, it is enough to take into consideration that $H^n(L) \equiv \mathrm{tr} L^n(w)$ is a Casimir function of $\mathfrak{g}$, hence:

$$\sum_{m,n=1}^{d} C^{(mn)}_{(ij),(kl)} L_{mn}(w) \frac{\partial H^n(L(w))}{\partial L_{kl}(w)} \equiv 0,$$

to expand $H^n(L(w))$ in the power series in the spectral parameter w, and to compare the summands with the equal degrees of w.

Hamiltonian equations and Lax type representation. Let us consider Hamiltonian equations of the form:

$$\frac{dl_{ij}^{(m)}}{dt_{s,K}^k} = \{l_{ij}^{(m)}, H_s^k(l_{kl}^{(m)})\}_K, \tag{12}$$

where $\{\ ,\ \}_K$ are the brackets $\{\ ,\ \}_0$ or $\{\ ,\ \}_{N+1}$ restricted to the subspace $\mathcal{M}_N$ and $t_{s,K}^k$ is "time" that corresponds to the Hamiltonian H_s^k and brackets $\{\ ,\ \}_K$. From item (iii) of Theorem 3, the next corollary follows:

Corollary 2. *Hamiltonian equations for the Hamiltonian H_s^m and the brackets $\{\ ,\}_0$ coincide with the Hamiltonian equations for the Hamiltonian H_{s+N+1}^m and brackets $\{\ ,\}_{N+1}$:*

$$\frac{\partial l_{ij}^{(m)}}{\partial t_{s,0}^k} = \{l_{ij}^{(m)}, H_s^k(l_{kl}^{(m)})\}_0 = \{H_{s+N+1}^k(l_{kl}^{(m)}), l_{ij}^{(n)}\}_{N+1}. \tag{13}$$

Let us re-write the Hamiltonian equations (12) in the Lax form. From the general considerations based on the Kostant—Adler scheme [2], it follows that these equations could be written in the Lax form:

$$\frac{dL(w)}{dt_{s,K}^k} = [L(w), M_{s,K}^k(w)], \tag{14}$$

where $L(w) \in \mathcal{M}_N$ and operator $M(w)$ is defined as follows: $M_{s,K}^k(w) = P_K \nabla_K H_s^k(L(w))|_{\mathcal{M}_N}$, where

$$\nabla_K H_s^k(L(w)) = \sum_{m=-\infty}^{\infty} \sum_{ij=1}^{d} \frac{\partial H_s^k}{\partial l_{ij}^{(m)}} X_{ij}^{-m+K} \tag{15}$$

is an algebra-valued gradient of H considered as a function on the whole space $\widetilde{\mathrm{so}(n)}^*_{\mathcal{H}_n}$, $K = 0$ or $K = N+1$ and $P_0 \equiv P_-$, $P_{N+1} \equiv P_+$, where $P_\pm$ are projection operators on the subalgebras $\widetilde{\mathrm{so}(n)}^\pm_{\mathcal{H}_n}$.

Now we can re-write Corollary 2 in the Lax form:

Corollary 3. *The following equality holds true:*

$$\frac{\partial L(w)}{\partial t_{s,0}^k} = [M_{s,0}^k, L(w)] = [L(w), M_{s+N+1,N+1}^k], \tag{16}$$

i.e "times" $t_{s,0}^k$ and $t_{s+N+1,N+1}^k$ coincide.

Remark 3. Corollary 3 means that, fixing the Lax matrix, we can write two different Lax representations for our Hamiltonian equations.

Remark 4. Corollary 3 is important from the point of view of applications: it gives us the possibility to choose one more simpler M operator from the two possible variants.

4. Integrable Systems in the Spaces of Small Quasigrade

The examples most interesting from the physical point of view usually arise in the spaces $\mathcal{M}_N$ with small N. Now we consider the case $N = 0$ and the algebras $\widetilde{\mathrm{so}(n)}_{\mathcal{H}'_n}$ in order to explicitly obtain generalized Clebsh and Neuman systems.

Remark 5. We will hereafter assume that $a_i \neq a_j$ for $i \neq j$. This requirement is necessary to guarantee the completeness of the family of mutually commuting integrals of motion.

Generalized Clebsh system — explicit construction. Let us consider a subspace $\mathcal{M}_0$. The corresponding Lax operator $L(w) \in \mathcal{M}_0$ has the following form:

$$L(w) = \sum_{i,j=1,n} l_{ij}^{(1)} \frac{y(w)}{w_i w_j} X_{ij}.$$

Commuting integrals are constructed using the expansions in the powers of w of the functions $H_k(w) = \mathrm{Tr}(L(w))^k$. We are especially interested in the quadratic Hamiltonians. Let us, for the purpose of convenience, introduce the following notation:

$$h(w) \equiv H^2(w) = \sum_{s=0}^{n-2} h_s(l_{ij}^{(1)}) w^s =$$

$$= \sum_{i,j=1}^{n} \prod_{k\neq i,j} (w - a_k)(l_{ij}^{(1)})^2.$$

We obtain

$$h_0 = (-1)^{n-2} \sum_{i,j=1}^{n} \frac{(\prod_{k=1}^{n} a_k)}{a_i a_j} (l_{ij}^{(1)})^2$$

$$h_1 = (-1)^{n-3} \sum_{i,j=1}^{n} \frac{(\prod_{k=1}^{n} a_k)}{a_i a_j} (l_{ij}^{(1)})^2 (\sum_{k\neq i,j} a_k^{-1})$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$h_{n-3} = (-1) \sum_{i,j=1}^{n} (\sum_{k=1}^{n} a_k - (a_i + a_j))(l_{ij}^{(1)})^2$$

$$h_{n-2} = \sum_{i,j=1}^{n} (l_{ij}^{(1)})^2.$$

In the space $\mathcal{M}_0$, two Poisson structures $\{\ ,\ \}_K$ exist which corresponds to the cases $K = 0, 1$. Let us consider the case $K = 1$ (case $K = 0$ corresponds to generalized tops and was considered in [8]):

$$\{l^{(1)}_{ij}, l^{(1)}_{kl}\}_1 = a_i\delta_{il}l^{(1)}_{kj} - a_j\delta_{kj}l^{(1)}_{il} + a_i\delta_{ik}l^{(1)}_{jl} - a_j\delta_{jl}l^{(1)}_{ki}. \tag{17}$$

The Lie algebraic structure, which is defined by these brackets, strongly depends on the constants a_i. Let us consider the case of the simplest "degeneration" $a_n \to 0$, $a_i \neq 0$, where $i < n$, that corresponds to the Lie algebra $\widetilde{so(n)}_{\mathcal{H}'_n}$. In this case, we have the following commutation relations:

$$\{l^{(1)}_{ij}, l^{(1)}_{kl}\}_1 = a_i\delta_{il}l^{(1)}_{kj} - a_j\delta_{kj}l^{(1)}_{il} + a_i\delta_{ik}l^{(1)}_{jl} - a_j\delta_{jl}l^{(1)}_{ki}$$

$$\{l^{(1)}_{ij}, l^{(1)}_{kn}\}_1 = a_i\delta_{ik}l^{(1)}_{jn} - a_j\delta_{kj}l^{(1)}_{in},$$

$$\{l^{(1)}_{in}, l^{(1)}_{jn}\}_1 = 0,$$

where $i, j, k < n$. Making the replacement of variables

$$m_{ij} = \frac{l^{(1)}_{ij}}{b_ib_j}, x_k = \frac{l^{(1)}_{kn}}{b_k}, \text{ where } b_i = a_i^{1/2}, i, j, k < n, \tag{18}$$

we obtain the standard commutation relations for the Lie algebra $e(n-1)$:

$$\{m_{ij}, m_{kl}\}_1 = \delta_{il}m_{kj} - \delta_{kj}m_{il} + \delta_{ik}m_{jl} - \delta_{jl}m_{ki},$$
$$\{m_{ij}, x_k\}_1 = \delta_{ik}x_k - \delta_{kj}x_i,$$
$$\{x_i, x_j\}_1 = 0.$$

Let us consider the explicit form of the above Hamiltonians in the limit $a_n \to 0$. By direct calculations, taking into account that, in the case $a_n = 0$, only some of the summands in the definition of each function h_k survive, we obtain:

$$h_0 = (-1)^{n-2}2(\prod_{k=1}^{n-1} a_k)\sum_{i=1}^{n-1}\frac{(l^{(1)}_{in})^2}{a_i}$$
$$h_1 = (-1)^{n-3}(\prod_{k=1}^{n-1} a_k)\sum_{i,j=1}^{n-1}(\frac{(l^{(1)}_{ij})^2}{a_ia_j} - 2\frac{(l^{(1)}_{in})^2}{a_i^2})$$
$$\dots\dots\dots\dots\dots$$
$$h_{n-3} = (-1)\sum_{i,j=1}^{n-1}(\sum_{k=1}^{n-1} a_k - (a_i + a_j))(l^{(1)}_{ij})^2 -$$
$$-2(\sum_{k=1}^{n-1} a_k - a_i)(l^{(1)}_{in})^2)$$
$$h_{n-2} = \sum_{i,j=1}^{n}(l^{(1)}_{ij})^2$$

For the case of the brackets $\{\ ,\ \}_1$ and $a_n = 0$, in order to have the above Hamiltonians in the standard $e(n-1)$ coordinates, we apply the coordinate transformation (18). As a result, we obtain:

$$h_0 = (-1)^{n-2}2(\prod_{k=1}^{n-1} a_k)\sum_{k=1}^{n-1} x_k^2$$
$$h_1 = (-1)^{n-3}(\prod_{k=1}^{n-1} a_k)(\sum_{i,j=1}^{n-1}(m_{ij}^2 - 2a_i^{-1}x_i^2) - h_0)$$
$$\dots\dots\dots\dots\dots$$
$$h_{n-3} = (\sum_{i,j=1}^{n-1} a_ia_j(a_i + a_j)m_{ij}^2 + 2a_i^2x_i^2) - (\sum_{k=1}^{n-1} a_k)h_{n-2}$$
$$h_{n-2} = (\sum_{i,j=1}^{n-1} a_ia_jm_{ij}^2 + 2a_ix_i^2).$$

These are the Hamiltonians of *generalized Clebsh systems.* In order to recognize this, we will consider a small-rank example.

Small-rank example: $e(3)$ Clebsh system. Let $n = 4$. In this case, we obtain the Clebsh integrable case of Kirchoff's system on $e(3)$. It was discussed in [4] and [6].

Hamiltonians of the Clebsh integrable case h_0, h_1, h_2, are already written in the standard form. Indeed, making the replacement of variables $m_k = \epsilon_{ijk}m_{ij}$, $x_k \equiv x_k$, in order to have standard $e(3)$ commutation relations,

$$\{m_i, m_j\} = \epsilon_{ijk}m_k,\ \{m_i, x_j\} = \epsilon_{ijk}x_k,\ \{x_i, x_j\} = 0,$$

we obtain

$$h_0 = (a_1a_2a_3)\sum_{i=1}^{3} x_i^2$$
$$h_1 = (a_1a_2a_3)(\sum_{i=1}^{3} m_i^2 - a_i^{-1}x_i^2) + (\sum_{i=1}^{3} a_i^{-1})h_0$$
$$h_2 = (\sum_{i}^{3}\frac{a_1a_2a_3}{a_i}m_i^2 + a_ix_i^2).$$

The function h_0 is a Casimir function. It is easy to see that the functions h_2 and h_1 (modulo Casimir h_0) are standard integrals of the $e(3)$ Clebsh system [6, 4].

Motion of a particle on the $(n-2)$-dimensional sphere in quadratic potential. In this paragraph, we will obtain Hamiltonians and integrals of motion of a particle on the n-dimensional sphere in quadratic potential. This system is known to be integrable [2]. It is usually called a generalized Neuman system. The

Hamiltonian of this system is given by the following formula:

$$H_{\{c_i\}}(p,x) = \sum_{i=1}^{n-1} p_i^2 + \sum_{i=1}^{n-1} c_i x_i^2,$$

where the coefficients c_i are arbitrary, coordinates x_i and momenta p_i are subjected to the constraints: $\sum\limits_{i=1}^{n-1} x_i^2 = r^2$ and $\sum\limits_{i=1}^{n-1} x_i p_i = 0$, and the Poisson brackets are canonical:

$$\{p_i, x_j\} = \delta_{ij},\ \{x_i, x_j\} = 0,\ \{p_i, p_j\} = 0.$$

We will show below that it could be considered as a special case of the Clebsh system and will obtain the correspondent hyperelliptic Lax pairs.

Let us consider the restriction of the Clebsh system on the degenerated coadjoint orbits of the group $E(n-1)$. More precisely, let us consider the degenerated orbit $O_{\min}$ of $E(n-1)$ of the minimal dimension. It is known that they coincide with T^*S^{n-2} [2]. It is also known that this orbit satisfies such constraints [9] that, after the restriction, elements m_{ij} could be parameterized as follows:

$$m_{ij} = x_i p_j - x_j p_i,\ \text{where}\ \sum_{i=1}^{n-1} x_i p_i = 0,\ \sum_{i=1}^{n-1} x_i^2 = r^2.$$

Let us consider the Hamiltonians of a Clebsh system, restricted on $o_{\min} = T^*S^{n-2}$:

$$h_0 = (-1)^{n-2}(\prod_{k=1}^{n-1} a_k)\sum_{k=1}^{n-1} x_k^2 = (\prod_{k=1}^{n-1} a_k) r^2,$$

$$h_1 = (-1)^{n-3}(\prod_{k=1}^{n-1} a_k)(\sum_{i,j=1}^{n-1}(x_i p_j - x_j p_i)^2 - a_i^{-1} x_i^2) - -(\sum_{k=1}^{n-1} a_k^{-1}) h_0,$$

....

$$h_{n-3} = (\sum_{i,j=1}^{n-1} a_i a_j (a_i + a_j)(x_i p_j - x_j p_i)^2 + a_i^2 x_i^2) - -(\sum_{k=1}^{n-1} a_k) h_{n-2},$$

$$h_{n-2} = (\sum_{i,j=1}^{n-1} a_i a_j (x_i p_j - x_j p_i)^2 + a_i x_i^2).$$

Taking into account that

$$\sum_{i,j=1}^{n-1}(x_i p_j - x_j p_i)^2 = 2(\sum_{i=1}^{n-1} x_i^2)(\sum_{j=1} p_j^2) - 2(\sum_{i=1}^{n-1} x_i p_i)^2$$

Now, taking into account the constraints $\sum\limits_{i=1}^{n-1} x_i^2 = r^2$ and $\sum\limits_{i=1}^{n-1} x_i p_i = 0$, we obtain

$$\sum_{i,j=1}^{n-1} m_{ij}^2 = 2r^2(\sum_{j=1} p_j^2).$$

This implies the Hamiltonian h_1 in the next form:

$$h_1 = (-1)^{n-1}(\prod_{k=1}^{n-1} a_k)(\sum_{i,j=1}^{n-1} 2r^2 p_i^2 - a_i^{-1} x_i^2) + c.$$

Now it is evident that modulo a constant, the following identity is true $h_1 = 2r^2(\prod\limits_{k=1}^{n-1}(-a_k)) H_{\{c_i\}}$, where $c_i = -(2r^2 a_i)^{-1}$, i.e., that we indeed obtain a generalized Neuman's Hamiltonian.

New Lax-type representations for the generalized Clebsh and Neuman systems. In this subsection, we will explicitly write a new $L-M$ pair for the generalized Clebsh and Neuman systems.

We begin with the case of a Clebsh system, considering a Neuman system as its special reduction. As follows directly from the results of the previous section, the L operator for a Clebsh system has the following form:

$$L(w) = \sum_{i<j}^{n-1} \frac{a_i^{1/2} a_j^{1/2} y(w)}{w_i w_j} m_{ij} X_{ij} + \sum_{i=1}^{n-1} \frac{a_i^{1/2} y(w)}{w_i w^{1/2}} x_i X_{in}. \tag{19}$$

Using Corollary 3 we obtain that the Lax equations for the Hamiltonian H_1^2 and brackets $\{\ ,\ \}$ can be written in two equivalent forms:

$$\frac{\partial L(w)}{\partial t_{1,1}^2} = [L(w), M_{1,1}^2] = [M_{0,0}^2, L(w)]. \tag{20}$$

Due to the fact that the operator $M_{0,0}^2$ has a simpler form, we will use it as the M operator of the Clebsh system. Direct calculation yields the following result:

$$M(w) \equiv M_{0,0}^2 = 2(\prod_{k=1}^{n-1} a_k)\sum_{i=1}^{n-1} \frac{x_i}{a_i^{1/2}} \frac{w_i}{w^{1/2}} X_{in}. \tag{21}$$

The $L-M$ pair for the Neuman system coincides with the $L-M$ pair for the Clebsh system restricted onto T^*S^{n-1}, i.e., one has to put $m_{ij} \equiv x_i p_j - x_j p_i$ into the $L-M$ pair of the Clebsh system (19).

Several remarks on the spectral curve. At the end of the paper, we want to make several comments on the algebraic curve, on whose Jacobian the equations of motion of the considered systems becomes linear.

Due to the standard formalism [10], the correspondent curve $R_{N,n}$ could be defined via the equation

$$R_{N,n}(w,u) \equiv \det(L(w) - u \cdot 1_n) = 0, \qquad (22)$$

where $L(w) \in \mathcal{M}_N$. It is necessary only to show, that the corresponding equation is indeed algebraic, i.e., that the function $R_{N,n}(w,u)$ is a polynomial in u and w:

Proposition 5. *Function $R_{N,n}$ is a polynomial in u and w of the following type:*

$$R_{N,n}(w,u) = \sum_{i,j} R^{ij}_{N,n}(H^k_m) u^i w^j,$$

where $i \in \overline{0,n}$, $j \in \overline{0,[n/2](N+n-2)}$.

Proof. Proposition 5 is proved by the direct calculation. Indeed, expanding $R_{N,n}(w,u)$ in the power series in the parameter u, we obtain: $R_{N,n}(w,u) = \sum\limits_{k=0}^{n} p_k u^{n-k}$, where p_k are the coefficients of the characteristic polynomial. On the other hand, it is well known that $p_k = -1/k(S_k + \sum\limits_{i=0}^{k-1} S_{k-i} p_i)$, where $S_k \equiv TrL(w)^k$. Due to the fact that the matrix $L(w)$ is skew, we, moreover, obtain that $p_{2k+1} = S_{2k+1} = 0$, and $p_{2k} = p_{2k}(S_2, ..., S_{2k})$. As follows from the definition of $L(w)$, $TrL(w)^{2k}$ is a polynomial in w of the degree $k(N+n-2)$, and its coefficients are expressed via Hamiltonians H^k_m.

Proposition 5 is proved.

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1. *Reyman A.G., Semenov Tian-Shansky M. A.* // Invent. Math.– 1979.–**54**.– P. 81–100

2. *Reyman A.G., Semenov Tian-Shansky M. A.* //VINITI: Cont. Probl. Math. Fundamental Trends. – 1989. – **6**.– P. 145–147.

3. *Kostant B.*// Adv. Math.–1979.–**34**.– P. 195–338.

4. *Holod P.I.*// Doklady of the Academy of Sciences of USSR.– 1984.– **276**, N 5.– P. 5–8.

5. *Holod P.I.*// Theor. Math. Phys.– 1987.– **70**, N 1.– P. 18–29.

6. *Bobenko A.I.* // Funct. Anal. and Appl.– 1986.–**20**, N 1.– P. 64–65.

7. *Skrypnyk T.*//Proc. XXIII Intern. Colloquium on the Group Theoretical Methods in Physics, 2000; `e-print: nlin.SI-0010005`

8. *Skrypnyk T.*// J. Math Phys.– 2001.– **48**, N 9.

9. *Boyarsky A., Skrypnyk T.*//Usp. Mat. Nauk.– 2000.– **55**, N3.

10. *Dubrovin B. A., Krichever I.M., Novikov S.P.* //VINITI, Fundamental trends.– 1985.– **4**.– P.179–285.

СИСТЕМИ КЛЕБША ТА НЕЙМАНА ЗІ СПЕЦІАЛЬНИХ КВАЗІГРАДУЙОВАНИХ АЛГЕБР ЛІ НА КРИВИХ ВИЩИХ РОДІВ

Т.В. Скрипник

Р е з ю м е

Ми конструюємо спеціальні so(n)-значні алгебри Лі на алгебраїчних кривих вищих родів. Використовуючи їх, ми отримуємо нову теоретичну лі-інтерпретацію і нові представлення Лакса зі спектральним параметром для узагальнених систем Клебша та Неймана.

СИСТЕМЫ КЛЕБША И НЕЙМАНА ИЗ СПЕЦИАЛЬНЫХ КВАЗИГРАДУИРОВАННЫХ АЛГЕБР ЛИ НА КРИВЫХ ВЫСШИХ РОДОВ

Т.В. Скрыпнык

Р е з ю м е

Мы конструируем специальные so(n)-значные алгебри Ли на алгебраических кривых высших родов. Используя их, мы находим новую теоретическую ли-интерпретацию и представления типа Лакса со спектральным параметром для обобщенных систем Клебша и Неймана.