ON A q-ANALOGUE OF THE PENROSE TRANSFORM [1]

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In the framework of the theory of quantum groups and their homogeneous spaces we consider two geometric realizations for the ladder representation of the quantum group $SU(2, 2)$ and their intertwining linear transformation which is a q-analogue of the Penrose transform. Our results hint that a great deal of constructions specific for the theory of quasi-coherent G-sheaves admit non-commutative analogues.

1. Introduction

In the framework of the theory of quantum groups and their homogeneous spaces, we consider two geometric realizations for the quantum ladder representation, together with an intertwining linear transformation — the quantum Penrose transform.

In section 2, we supply a preliminary material on the classical Penrose transform and prove (1). The q-analogue of (1) is to be used in Section 3 to produce a quantum Penrose transform.

Our results hint that a great deal of constructions specific of the theory of quasi-coherent sheaves admit non-commutative analogues. This research is motivated by a possibility to use the results of non-commutative algebraic geometry for producing and studying Harish-Chandra modules over quantum universal enveloping algebras.

There is a plenty of literature on the Penrose transform, quantum groups, and non-commutative algebraic geometry. We restrict ourselves to mentioning monographs [2, 6, 3], papers [1, 10], and preprint [9].

Note that noncommutative analogues for the Penrose transform and covariant differential operators are also considered in preprints [8, 12] and in papers [5, 4, 7], respectively, in a completely different context.

2. The Classical Case

To recall the definition of the Penrose transform, we restrict ourselves to a simplest substantial example. In this special case, the Penrose transform intertwines the cohomology of the sheaf $\mathcal{O}(-2)$ on

$$U' = \{(u_1 : u_2 : u_3 : u_4) \in \mathbb{CP}^3 |\, u_3 \neq 0 \quad \text{or} \quad u_4 \neq 0\}$$

and sections of the sheaf $\mathcal{O}(-1)$ on some open affine submanifold of the Grassmann manifold $\mathrm{Gr}_2(\mathbb{C}^4) \hookrightarrow \mathbb{CP}^5$. Instead of the Grassmann manifold, we prefer to consider the Stifel manifold of ordered linear independent pairs of vectors in $\mathbb{C}^4$. In this context, the GL_2-covariant sections on the Stifel manifold work as sections of the sheaf $\mathcal{O}(-1)$ on $\mathrm{Gr}_2(\mathbb{C}^4)$.

To each matrix $\mathbf{t} = (t_{ij})_{i=1,2;j=1,2,3,4} \in \mathrm{Mat}_{2,4}$, associate the pairs of vectors in $\mathbb{C}^4$:

$$(t_{11}, t_{12}, t_{13}, t_{14}), \qquad (t_{21}, t_{22}, t_{23}, t_{24}).$$

Consider $U'' = \{\mathbf{t} \in \mathrm{Mat}_{2,4} |\, t_{13}t_{24} - t_{14}t_{23} \neq 0\}$. Every point $\mathbf{u} = (u_1 : u_2 : u_3 : u_4) \in U'$ determines a one-dimensional subspace $L_\mathbf{u} \subset \mathbb{C}^4$, and every point $\mathbf{t} \in U''$ determines a two-dimensional subspace $L_\mathbf{t}$ generated by the vectors of the corresponding pair. Let $U = \{(\mathbf{u}, \mathbf{t}) \in U' \times U'' |\, L_\mathbf{u} \subset L_\mathbf{t}\}$. We thus get a 'double fibration' $U' \underset{\eta}{\leftarrow} U \underset{\tau}{\rightarrow} U''$, which leads to the Penrose transform. It should be noted that every line $L_\mathbf{t}$ is of the form

$$L = \mathbb{C}(\zeta_1, \zeta_2) \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{pmatrix}, \qquad (\zeta_1, \zeta_2) \in \mathbb{C}^2.$$

Hence, the above double fibration is isomorphic to the double fibration

$$U' \underset{\pi}{\leftarrow} \mathbb{CP}^1 \times U'' \underset{\mathrm{pr}_2}{\rightarrow} U'',$$

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with

$$\pi:\left((\zeta_1:\zeta_2),\begin{pmatrix}t_{11}&t_{12}&t_{13}&t_{14}\\ t_{21}&t_{22}&t_{23}&t_{24}\end{pmatrix}\right)\mapsto$$

$$\mapsto((\zeta_1t_{11}+\zeta_2t_{21}):(\zeta_1t_{12}+\zeta_2t_{22}):(\zeta_1t_{13}+$$

$$+\zeta_2t_{23}):(\zeta_1t_{14}+\zeta_2t_{24})).$$

We thus get a coordinate description for the double fibration in question; this coordinate description is going to be implicit in all subsequent computations. Let us look at the cohomologies.

Consider an open affine cover $U'=U_1\cup U_2$,

$$U_1=\{(u_1:u_2:u_3:u_4)\in U'|\,u_3\neq 0\},$$
$$U_2=\{(u_1:u_2:u_3:u_4)\in U'|\,u_4\neq 0\},$$

and compute the Čech cohomology $\check{H}^1(U',\mathcal{O}(-2))$. Let $\mathbb{C}[u_1,u_2,u_3^{\pm1},u_4^{\pm1}]$ be the Laurent polynomials in indeterminates u_3, u_4, with coefficients from $\mathbb{C}[u_1,u_2]$. In a similar way,introduce $\mathbb{C}[u_1,u_2,u_3^{\pm1},u_4]$, $\mathbb{C}[u_1,u_2,u_3,u_4^{\pm1}]$; of course, these appear to be $U\mathfrak{sl}_4$-modules.

It follows from the definition of the Čech complex that there exists a natural isomorphism of $U\mathfrak{sl}_4$-modules:

$$\check{H}^1(U',\mathcal{O}(-2))=$$

$$=\{f\in\mathbb{C}[u_1,u_2,u_3^{\pm1},u_4^{\pm1}]\,/\big(\mathbb{C}[u_1,u_2,u_3^{\pm1},u_4]\;+$$

$$+\mathbb{C}[u_1,u_2,u_3,u_4^{\pm1}]\big)\big|\deg f=-2\}.$$

Hence, the Laurent polynomials

$$\frac{u_1^{j_1}u_2^{j_2}}{u_3^{j_3}u_4^{j_4}},\quad j_3\geq 1\quad\&\quad j_4\geq 1\quad\&\quad j_3+j_4=j_1+j_2+2,$$

form a basis of the vector space $H^1(U',\mathcal{O}(-2))$.

Consider the trivial bundle over U'' with fiber $H^1(\mathbb{CP}^1,\mathcal{O}(-2))$. It is known that $H^1(\mathbb{CP}^1,\mathcal{O}(-2))\simeq\mathbb{C}$, and the isomorphism is available via choosing an open affine cover $\mathbb{CP}^1=\{(\zeta_1:\zeta_2)|\,\zeta_1\neq 0\}\cup\{(\zeta_1:\zeta_2)|\,\zeta_2\neq 0\}$. Specifically, $\sum\limits_{j+k=-2}c_{jk}\zeta_1^j\zeta_2^k\mapsto c_{-1,-1}$. In a different notation, $f\mapsto\mathrm{CT}(\zeta_1\zeta_2f)$, with $\mathrm{CT}:\sum\limits_{j,k}c_{jk}\zeta_1^j\zeta_2^k\mapsto c_{00}$ (the constant term of a series). Now $\mathcal{P}f$ is defined as a higher direct image of the cohomology class η^*f: $\mathcal{P}f=\tau_*^1\eta^*f$. The linear map τ_*^1 is called the integration along the fibers of τ. We restrict ourselves to computing this 'integral' inside the infinitesimal neighborhood of $\mathbf{t}_0=\begin{pmatrix}0&0&0&1\\0&0&1&0\end{pmatrix}$ by using formal series in t_{11}, t_{12}, t_{13}, t_{14}^{-1}, t_{21}, t_{23}^{-1}, t_{24} with coefficients from $\mathbb{C}[\zeta_1^{\pm1},\zeta_2^{\pm1}]$. Of course, $\eta^*:f(\boldsymbol{u})\mapsto f(\boldsymbol{\zeta t})$. So, in the coordinate description

$$\begin{aligned}&\mathcal{P}\;:\;f(\boldsymbol{u})\mapsto CT_{\boldsymbol{\zeta}}(\zeta_1\zeta_2f(\boldsymbol{\zeta t})),\\ &f\;\in\;u_3^{-1}u_4^{-1}\mathbb{C}[u_1,u_2,u_3^{-1},u_4^{-1}],\end{aligned}\tag{1}$$

with $\mathrm{CT}_{\boldsymbol{\zeta}}$ being the constant term in the indeterminate $\boldsymbol{\zeta}$. EXAMPLE. Compute $\mathcal{P}(1/(u_3u_4))$. One has:

$$\frac{1}{\zeta_1t_{13}+\zeta_2t_{23}}=\frac{1}{\zeta_2t_{23}}\sum_{i=0}^{\infty}(-1)^i\left(\frac{\zeta_1t_{13}}{\zeta_2t_{23}}\right)^i,$$

$$\frac{1}{\zeta_1t_{14}+\zeta_2t_{24}}=\frac{1}{\zeta_1t_{14}}\sum_{j=0}^{\infty}(-1)^j\left(\frac{\zeta_2t_{24}}{\zeta_1t_{14}}\right)^j.$$

Hence,

$$\mathcal{P}\left(\frac{1}{u_3u_4}\right)=$$

$$=CT_{\boldsymbol{\zeta}}\left(\frac{1}{t_{23}t_{14}}\sum_{i,j=0}^{\infty}(-1)^{i+j}\left(\frac{\zeta_1t_{13}}{\zeta_2t_{23}}\right)^i\left(\frac{\zeta_2t_{24}}{\zeta_1t_{14}}\right)^j\right)=$$

$$=\frac{1}{t_{23}t_{14}}\sum_{k=0}^{\infty}\left(\frac{t_{13}t_{24}}{t_{23}t_{14}}\right)^k=\frac{1}{t_{23}t_{14}}\cdot\frac{1}{1-\frac{t_{13}t_{24}}{t_{23}t_{14}}}=$$

$$=-\frac{1}{t_{13}t_{24}-t_{14}t_{23}}.$$

REMARK. It is known that the Penrose transform is an isomorphism between the two realizations for the 'ladder' representation of $\mathfrak{sl}_4$: the representation in $H^1(U',\mathcal{O}(-2))$ and the representation in

$$\{\psi(z_1^1,z_2^1,z_1^2,z_2^2)(t_{13}t_{24}-t_{14}t_{23})^{-1}\in$$

$$\in H^0(U'',\mathcal{O}(-1))|\,\Box\psi=0\},$$

where $\Box\psi\overset{\text{def}}{=}\dfrac{\partial^2\psi}{\partial z_1^1\partial z_2^2}-\dfrac{\partial^2\psi}{\partial z_2^1\partial z_1^2}$, and

$$z_1^1=\frac{t_{11}t_{23}-t_{13}t_{21}}{t_{13}t_{24}-t_{14}t_{23}},z_2^1=\frac{t_{12}t_{23}-t_{13}t_{22}}{t_{13}t_{24}-t_{14}t_{23}}$$

$$z_1^2=\frac{t_{11}t_{24}-t_{14}t_{21}}{t_{13}t_{24}-t_{14}t_{23}},z_2^2=\frac{t_{12}t_{24}-t_{14}t_{22}}{t_{13}t_{24}-t_{14}t_{23}}.$$

The vectors $1/(u_3u_4)$ and $1/(t_{13}t_{24}-t_{14}t_{23})$ are lowest weight vectors for the above representations of $\mathfrak{sl}_4$. Of course, z_1^1, z_2^1, z_1^2, z_2^2 can be considered as the standard coordinates on the big cell $t_{13}t_{24}-t_{14}t_{23}\neq 0$ of the Grassmanian $\mathrm{Gr}_2(\mathbb{C}^4)$.

3. The Quantum Case

In the previous section, we have produced formula (1) which can be treated as a definition of the Penrose transform in the classical case. Now our intention is to produce a q-analogue of (1). The principal difference from the constructions of Section 2 is in replacement of the functors η^*, τ_*^1 of the sheaf theory with the corresponding morphisms of $U_q\mathfrak{sl}_4$-modules. Here, $U_q\mathfrak{sl}_4$ is a quantum universal enveloping algebra. It is a Hopf algebra over the ground field $\mathbb{C}(q)$ and is determined by the generators $\{E_i,F_i,K_i^{\pm1}\}_{i=1,2,3}$ and the well-known Drinfeld-Jimbo relations [6].)

The quantum projective space $\mathbb{CP}^3_{\mathrm{quant}}$ is defined in terms of a $\mathbb{Z}_+$-graded algebra $\mathbb{C}[u_1,u_2,u_3,u_4]_q$ whose generators u_1, u_2, u_3, u_4 are subject to the commutation relations

$$u_iu_j=qu_ju_i,\qquad i<j.$$

Just as in the classical case, $\deg u_k=1$, $k=1,2,3,4$. The localization $\mathbb{C}[u_1,u_2,u_3^{\pm1},u_4^{\pm1}]_q$ of $\mathbb{C}[u_1,u_2,u_3,u_4]_q$ with respect to the multiplicative system $(u_3u_4)^{\mathbb{N}}$ is equipped in a standard way with a structure of $U_q\mathfrak{sl}_4$-module algebra. The subalgebras $\mathbb{C}[u_1,u_2,u_3^{\pm1},u_4]_q$, $\mathbb{C}[u_1,u_2,u_3,u_4^{\pm1}]_q$ constitute $U_q\mathfrak{sl}_4$-submodules of the $U_q\mathfrak{sl}_4$-module $\mathbb{C}[u_1,u_2,u_3^{\pm1},u_4^{\pm1}]_q$. Thus, we come to

$$V'=\mathbb{C}[u_1,u_2,u_3^{\pm1},u_4^{\pm1}]_q/\big(\mathbb{C}[u_1,u_2,u_3^{\pm1},u_4]_q+$$
$$+\mathbb{C}[u_1,u_2,u_3,u_4^{\pm1}]_q\big)\tag{2}$$

as a q-analogue of the $U\mathfrak{sl}_4$-module $H^1(U',\mathcal{O}(-2))$.

We have produced a q-analogue for the first geometric realization of the 'ladder representation'. Turn to a construction of its second geometric realization.

The algebra $\mathbb{C}[\mathrm{Mat}_{2,4}]_q$ of polynomials on the quantum matrix space is determined by its generators $\{t_{ij}\}_{i=1,2;j=1,2,3,4}$ and the well-known commutation relations

$$\begin{aligned}&t_{ik}t_{jk}=qt_{jk}t_{ik},\quad t_{ki}t_{kj}=qt_{kj}t_{ki}, && i<j,\\ &t_{ij}t_{kl}=t_{kl}t_{ij}, && i<k,\quad j>l,\\ &t_{ij}t_{kl}-t_{kl}t_{ij}=(q-q^{-1})t_{ik}t_{jl}, && i<k,\quad j<l.\end{aligned}$$

The element $t=t_{13}t_{24}-qt_{14}t_{23}$ quasi-commutes with all the generators t_{ij}, $i=1,2$, $j=1,2,3,4$. Let $\mathbb{C}[\mathrm{Mat}_{2,4}]_{q,t}$ be a localization of $\mathbb{C}[\mathrm{Mat}_{2,4}]_q$ with respect to the multiplicative system $t^{\mathbb{N}}$ and $U_q\mathfrak{sl}_2$ the quantum universal enveloping algebra (determined by the generators E, F, $K^{\pm1}$ and the Drinfeld-Jimbo relations).

$\mathbb{C}[\mathrm{Mat}_{2,4}]_{q,t}$ is equipped in a standard way with a structure of $U_q\mathfrak{sl}_2\otimes U_q\mathfrak{sl}_4$-module algebra. In particular, $\mathbb{C}[\mathrm{Mat}_{2,4}]_{q,t}$ is a $U_q\mathfrak{sl}_4$-module algebra.

Introduce the notation:

$$\begin{aligned}&z_1^1=t^{-1}(t_{11}t_{23}-qt_{13}t_{21}),\\ &z_2^1=t^{-1}(t_{12}t_{23}-qt_{13}t_{22}),\\ &z_1^2=t^{-1}(t_{11}t_{24}-qt_{14}t_{21}),\\ &z_2^2=t^{-1}(t_{12}t_{24}-qt_{14}t_{22}).\end{aligned}$$

It is well known and easily deducible that

$$\begin{aligned}&z_k^iz_k^j=qz_k^jz_k^i,\quad z_i^kz_j^k=qz_j^kz_i^k, && i<j,\\ &z_j^iz_l^k=z_l^kz_j^i, && i<k,\quad j>l,\\ &z_j^iz_l^k-z_l^kz_j^i=(q-q^{-1})z_k^iz_l^j, && i<k,\quad j<l.\end{aligned}$$

It follows that the subalgebra generated by z_1^1, z_2^1, z_1^2, z_2^2 is 'canonically' isomorphic to the algebra $\mathbb{C}[\mathrm{Mat}_{2,2}]_q$ of 'polynomials on the quantum matrix space'. It is easy to demonstrate that $\mathbb{C}[\mathrm{Mat}_{2,2}]_qt^{-1}$ is a $U_q\mathfrak{sl}_4$-submodule of the $U_q\mathfrak{sl}_4$-module $\mathbb{C}[\mathrm{Mat}_{2,4}]_{q,t}$.

The simple submodule of the $U_q\mathfrak{sl}_4$-module $\mathbb{C}[\mathrm{Mat}_{2,2}]_qt^{-1}$ we are interested in is distinguished via a q-analogue $\Box_q$ of the wave operator $\Box$:

$$\Box_q=\frac{\partial}{\partial z_1^1}\frac{\partial}{\partial z_2^2}-q\frac{\partial}{\partial z_2^1}\frac{\partial}{\partial z_1^2}.$$

Specifically, $V''=\{\psi t^{-1}|\,\Box_q\psi=0,\,\psi\in\mathbb{C}[\mathrm{Mat}_{2,2}]_q\}$. It is worth to note that the operators $\dfrac{\partial}{\partial z_j^i}$ are defined in terms of a $U_q\mathfrak{sl}_4$-invariant first order differential calculus in $\mathbb{C}[\mathrm{Mat}_{2,2}]_q$:

$$df=\sum_{i,j}\frac{\partial f}{\partial z_j^i}dz_j^i.$$

In turn, this first order differential calculus is defined by the following well-known 'commutation' relations:

$$\begin{cases} z_1^1 dz_1^1 = q^{-2} dz_1^1 \cdot z_1^1 \\ z_1^1 dz_2^1 = q^{-1} dz_2^1 \cdot z_1^1 \\ z_1^1 dz_1^2 = q^{-1} dz_1^2 \cdot z_1^1 \\ z_1^1 dz_2^2 = dz_2^2 \cdot z_1^1 \end{cases}$$

$$\begin{cases} z_2^1 dz_1^1 = q^{-1} dz_1^1 \cdot z_2^1 + (q^{-2}-1) dz_2^1 \cdot z_1^1 \\ z_2^1 dz_2^1 = q^{-2} dz_2^1 \cdot z_2^1 \\ z_2^1 dz_1^2 = dz_1^2 \cdot z_2^1 + (q^{-1}-q) dz_2^2 \cdot z_1^1 \\ z_2^1 dz_2^2 = q^{-1} dz_2^2 \cdot z_2^1 \end{cases}$$

$$\begin{cases} z_1^2 dz_1^1 = q^{-1} dz_1^1 \cdot z_1^2 + (q^{-2}-1) dz_1^2 \cdot z_1^1 \\ z_1^2 dz_2^1 = dz_2^1 \cdot z_1^2 + (q^{-1}-q) dz_2^2 \cdot z_1^1 \\ z_1^2 dz_1^2 = q^{-2} dz_1^2 \cdot z_1^2 \\ z_1^2 dz_2^2 = q^{-1} dz_2^2 \cdot z_1^2 \end{cases}$$

$$\begin{cases} z_2^2 dz_1^1 = dz_1^1 \cdot z_2^2 + (q^{-1}-q) dz_2^1 \cdot z_1^2 + \\ \qquad +(q^{-1}-q) dz_1^2 \cdot z_2^1 + (q^{-1}-q)^2 dz_2^2 \cdot z_1^1 \\ z_2^2 dz_2^1 = q^{-1} dz_2^1 \cdot z_2^2 + (q^{-2}-1) dz_2^2 \cdot z_2^1 \\ z_2^2 dz_1^2 = q^{-1} dz_1^2 \cdot z_2^2 + (q^{-2}-1) dz_2^2 \cdot z_1^2 \\ z_2^2 dz_2^2 = q^{-2} dz_2^2 \cdot z_2^2 \end{cases}$$

We thus get two $U_q\mathfrak{sl}_4$-modules V', V''; our intention is to find an explicit form of the linear map which provides an isomorphism $\mathcal{P} : V' \to V''$.

We follow the ideas of classical constructions described in Section 2 in considering the quantum projective space $\mathbb{CP}^1_{\mathrm{quant}}$. More precisely, let us consider a $\mathbb{Z}_+$-graded algebra $\mathbb{C}[\zeta_1, \zeta_2]_q$:

$$\zeta_1\zeta_2 = q\zeta_2\zeta_1, \qquad \deg(\zeta_1) = \deg(\zeta_2) = 1,$$

together with its localization $\mathbb{C}[\zeta_1^{\pm 1}, \zeta_2^{\pm 1}]_q$ with respect to the multiplicative system $(\zeta_1\zeta_2)^{\mathbb{N}}$. The algebra $\mathbb{C}[\zeta_1^{\pm 1}, \zeta_2^{\pm 1}]_q$ is equipped in a standard way with a structure of $U_q\mathfrak{sl}_2$-module algebra. The following homomorphism of algebras will work as the operator $f(\boldsymbol{u}) \mapsto f(\boldsymbol{\zeta t})$:

$$\eta^* : \mathbb{C}[u_1, u_2, u_3, u_4]_q \to \mathbb{C}[\zeta_1, \zeta_2]_q \otimes \mathbb{C}[\mathrm{Mat}_{2,4}]_q,$$
$$\eta^* : u_j \mapsto \zeta_1 \otimes t_{1j} + \zeta_2 \otimes t_{2j}, \qquad j = 1, 2, 3, 4.$$

To follow the constructions of Section 2, we have to invert the elements $\zeta_1 \otimes t_{13} + \zeta_2 \otimes t_{23}$, $\zeta_1 \otimes t_{14} + \zeta_2 \otimes t_{24}$ in a suitable localization of $\mathbb{C}[\zeta_1^{\pm 1}, \zeta_2^{\pm 1}]_q \otimes \mathbb{C}[\mathrm{Mat}_{2,4}]_{q,t}$. It is easy to verify that

$$(t_{14}t_{23})^2 \cdot \mathbb{C}[\mathrm{Mat}_{2,4}]_{q,t} \subset \mathbb{C}[\mathrm{Mat}_{2,4}]_{q,t} \cdot (t_{14}t_{23}),$$
$$\mathbb{C}[\mathrm{Mat}_{2,4}]_{q,t} \cdot (t_{14}t_{23})^2 \subset (t_{14}t_{23}) \cdot \mathbb{C}[\mathrm{Mat}_{2,4}]_{q,t}.$$

Thus, we have a well-defined localization of $\mathbb{C}[\mathrm{Mat}_{2,4}]_{q,t}$ with respect to the multiplicative system $(t_{14}t_{23})^{\mathbb{N}}$. In an appropriate completion of this algebra, one has the following relations:

$$(\zeta_1 \otimes t_{13} + \zeta_2 \otimes t_{23})^{-1} =$$

$$= (\zeta_2 \otimes t_{23})^{-1} \sum_{i=0}^{\infty} (-1)^i \left(\zeta_1\zeta_2^{-1}\right)^i \otimes \left(t_{13}t_{23}^{-1}\right)^i,$$

$$(\zeta_1 \otimes t_{14} + \zeta_2 \otimes t_{24})^{-1} =$$

$$= \left(\sum_{j=0}^{\infty} (-1)^j \left(\zeta_1^{-1}\zeta_2\right)^j \otimes \left(t_{14}^{-1}t_{24}\right)^j \right) (\zeta_1 \otimes t_{14})^{-1}.$$

We define the quantum Penrose transform by

$$\mathcal{P}_q f = (\mathrm{CT} \otimes \mathrm{id})(\zeta_1\zeta_2 \otimes 1)(\eta^* f),$$

where, just as above, $\mathrm{CT} : \sum_{ij} c_{ij}\zeta_1^i\zeta_2^j \mapsto c_{-1,-1}$, and f belongs to the linear span of the elements

$$u_1^{j_1} u_2^{j_2} u_3^{-j_3} u_4^{-j_4}, \quad j_3 \geq 1 \ j_4 \geq 1 \ j_1 + j_2 - j_3 - j_4 = -2. \tag{3}$$

Now (3.) determines a $U_q\mathfrak{sl}_4$-module structure in this linear span since monomials (3) form a basis in the vector space V'.

APPENDIX

We sketch here the proof of the fact that $\mathcal{P}_q$ is an isomorphism of $U_q\mathfrak{sl}_4$-modules $V' \xrightarrow{\sim} V''$.

It follows from the definition that $\mathcal{P}_q$ is a morphism of $U_q\mathfrak{sl}_4$-modules. In view of the simplicity of V' and V'', it suffices to prove that $\mathcal{P}_q$ takes the (lowest weight) vector $u_3^{-1}u_4^{-1} \in V'$ to the (lowest weight) vector $-(t_{13}t_{24} - qt_{14}t_{23})^{-1}$. We start with an auxiliary statement:

$$\left(1 - \left(t_{23}^{-1}t_{13}\right)\left(t_{14}^{-1}t_{24}\right)\right)^{-1} = \sum_{k=0}^{\infty} q^{-2k} \left(t_{23}^{-1}t_{13}\right)^k \left(t_{14}^{-1}t_{24}\right)^k. \tag{4}$$

It follows from the commutation relation

$$\left(t_{14}^{-1}t_{24}\right)\left(t_{23}^{-1}t_{13}\right) = q^{-2}\left(t_{23}^{-1}t_{13}\right)\left(t_{14}^{-1}t_{24}\right) + 1 - q^{-2} \tag{5}$$

and relation (6.5) of [11].

An application of (4) allows one to prove that

$$\mathcal{P}_q\left(u_3^{-1}u_4^{-1}\right) = -(t_{13}t_{24} - qt_{14}t_{23})^{-1}.$$

In fact,

$$\mathcal{P}_q\left(u_3^{-1}u_4^{-1}\right) = \mathrm{CT}\otimes\mathrm{id}\Bigg(\zeta_2^{-1}\otimes t_{23}^{-1}\times$$

$$\times\left(\sum_{i=0}^{\infty}(-1)^i(\zeta_1\zeta_2^{-1})^i\otimes q^{-i}\left(t_{23}^{-1}t_{13}\right)^i\right)\times$$

$$\times\left(\sum_{j=0}^{\infty}(-1)^j(\zeta_1\zeta_2^{-1})^j\otimes\left(t_{14}^{-1}t_{24}\right)^j\right)\zeta_1^{-1}\otimes t_{14}^{-1}\Bigg).$$

On the other hand,

$$\zeta_2^{-1}\left(\zeta_1\zeta_2^{-1}\right)^k\left(\zeta_1^{-1}\zeta_2\right)^k\zeta_1^{-1} = q^{-k}\zeta_2^{-1}\zeta_1^{-1} = q^{-k-1}\zeta_1^{-1}\zeta_2^{-1}.$$

Hence,

$$\mathcal{P}_q\left(u_3^{-1}u_4^{-1}\right) = q^{-1}t_{23}^{-1}\left(\sum_{k=0}^{\infty}q^{-2k}\left(t_{14}^{-1}t_{24}\right)^k\left(t_{23}^{-1}t_{13}\right)^k\right)t_{14}^{-1} =$$

$$= q^{-1}\left(t_{14}\left(1-t_{23}^{-1}t_{13}t_{14}^{-1}t_{24}\right)t_{23}\right)^{-1} = -(t_{13}t_{24}-qt_{14}t_{23})^{-1}, \quad (6)$$

which completes the proof.

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ПРО q-АНАЛОГ ПЕРЕТВОРЕННЯ ПЕНРОУЗА

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Р е з ю м е

В рамках теорії квантових груп та їх однорідних просторів розглянуто дві реалізації драбинчастого представлення квантової групи $SU(2,\,2)$ та сплітаючий їх лінійний оператор, що є q-аналогом перетворення Пенроуза. Отримані результати вказують на те, що багато конструкцій теорії квазікогерентних G-пучків мають некомутативні аналоги.

О q-АНАЛОГЕ ПРЕОБРАЗОВАНИЯ ПЕНРОУЗА

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Р е з ю м е

В рамках теории квантовых групп и их однородных пространств рассматриваются две реализации лестничного представления квантовой группы $SU(2,\,2)$ и сплетающий их линейный оператор, являющийся q-аналогом преобразования Пенроуза. Наши результаты указывают на то, что многие конструкции теории квазикогерентных G-пучков имеют некоммутативные аналоги.