A METHOD OF ADDITIONAL STRUCTURES ON THE OBJECTS OF A CATEGORY AS A BACKGROUND FOR CATEGORY ANALYSIS IN PHYSICS[1]

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In this article, we follow from a separate object to the group of its automorphisms and its invariants, then to category of such objects and functors on it. Thus, from the theory of invariants, we go to the theory of equivariant maps and then to the problem of the description functors from one category in other. As was written by Herman Weyl, "F. Klein has realized and applied a group as the great organizing and simplifying principle in Algebra, Geometry and Analysis." Today this mission of Groups goes to Categories.

1. Introduction

Category theory provides a compact method of encoding mathematical structures in a uniform way, thereby enabling the use of general theorems on, for example, equivalence and universal constructions. Any geometric structure (including the invariant structures of physics and calculus) is determined in some category of fiber bundle spaces associated with the given principal bundle over some basic manifold. In the frames of F. Klein program, the basic manifold is a homogeneous (as usual), G-space and the space of stratification — the group G by itself is stratified with respect to the residue classes. In general case it is advisable to consider the acting of a pseudogroup of some local diffeomorphisms on the basic manifold. This action can be lifted to the principal bundle up to the action of the pseudogroup of all local automorphisms of this bundle. In both cases, the set of all bundles associated with the original principal bundle forms the category which morphisms are G-scopes.

Definition 1.1. *A category is a quadruple* $(\mathrm{Ob}, \mathrm{Hom}, \mathrm{id}, \circ)$ *consisting of:*
(C1) a class Ob *of objects;*
(C2) a set $\mathrm{Hom}(A,B)$ *of morphisms for each ordered pair* (A,B) *of objects;*
(C3) a morphism $\mathrm{id}_A \in \mathrm{Hom}(A,A)$*; for each object* A*, the identity of* A*;*
(C4) a composition law associating a morphism $g\circ f \in \mathrm{Hom}(A,C)$ *to each pair of morphisms* $f \in \mathrm{Hom}(A,B)$ *and* $g \in \mathrm{Hom}(B,C)$*;*
which is such that:
(M1) $h\circ(g\circ f) = (h\circ g)\circ f$ *for all* $f \in \mathrm{Hom}(A,B)$, $g \in \mathrm{Hom}(B,C)$, *and* $h \in \mathrm{Hom}(C,D)$*;*
(M2) $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ *for all* $f \in Hom(A,B)$*;*
(M3) the sets $\mathrm{Hom}(A,B)$ *are pairwise disjoint.*

This last axiom is necessary so that given a morphism we can identify its domain A and codomain B, however it can always be satisfied by replacing $\mathrm{Hom}(A,B)$ by the set $\mathrm{Hom}(A,B)\times(\{A\},\{B\})$.

Examples

1.1. The classic example is **Sets**, the category with sets as objects and functions as morphisms, and the usual composition of functions as a composition. But lots of the time *in mathematics* one is some category or other, e.g.:

$\mathbf{Vect}_k$ — vector spaces over a field k as objects; k-linear maps as morphisms;

Group — groups as objects, homomorphisms as morphisms;

Top — topological spaces as objects, continuous functions as morphisms;

Diff — smooth manifolds as objects, smooth maps as morphisms;

Ring — rings as objects, ring homomorphisms as morphisms;

or *in physics*:

Symp — symplectic manifolds as objects, symplectomorphisms as morphisms;

Poiss — Poisson manifolds as objects, Poisson maps as morphisms;

Hilb — Hilbert spaces as objects, unitary operators as morphisms.

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1.2. The typical way to think about symmetry is related to the concept of a "group". But to get a concept of symmetry that's really up to the demands put on it by modern mathematics and physics, we need — at the very least — to work with a "category" of symmetries, rather than a group of symmetries.

To see this, first ask: what is a category with one object? It is a — "*monoid*". The "usual" definition of a monoid is as follows: a set M with an associative binary product and a unit element 1 such that $al = la = a$ for all a in M. Monoids abound in mathematics; they are, in a sense, the most primitive interesting algebraic structures.

To check that a category with one object is "essentially just a monoid", note that if our category C has one object x, the set $\mathrm{Hom}(x, x)$ of all morphisms from x to x is indeed a set with an associative binary product, namely composition, and a unit element, namely id_x.

How about categories in which every morphism is invertible? We say a morphism $f : x \to y$ in a category has inverse $g : y \to x$ if $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. Well, a category in which every morphism is invertible is called a "groupoid".

Finally, a group is a category with one object in which every morphism is invertible. It's both a monoid and a groupoid!

When we use groups in physics to describe symmetry, we think of each element g of the group G as a "process". The element 1 corresponds to the "process of doing nothing at all". We can compose processes g and h — do h and then g — and get the product $g \circ h$. Crucially, every process g can be "undone" using its inverse g^{-1}.

So: a monoid is like a group, but the "symmetries" no longer need be invertible; a category is like a monoid, but the "symmetries" no longer need to be composable.

1.3. The operation of "evolving initial data from one spacelike slice to another" is a good example of a "partially defined" process: it only applies to initial data on that particular spacelike slice. So the dynamics in special or general relativity is most naturally described *using groupoids*. Only after pretending that all the spacelike slices are the same can we pretend we are using a group. It is very common to pretend that groupoids are groups, since groups are more familiar, but often insight is lost in the process. Also, one can only pretend a groupoid is a group if all its objects are isomorphic. Groupoids really are more general.

So: in contrast to a set, which consists of a static collection of "things", a category consists not only of objects or "things" but also morphisms which can viewed as "processes" transforming one thing into another. Similarly, in a 2-category, the 2-morphisms can be regarded as "processes between processes", and so on. The eventual goal of basing mathematics upon omega-categories is thus to allow us the freedom to think of any process as the sort of thing higher-level processes can go between. By the way, it should also be very interesting to consider "$\mathbb{Z}$-categories" (where $\mathbb{Z}$ denotes the integers), having j-morphisms not only for $j = 0, 1, 2, ...$ but also for negative j. Then we may also think of any thing as a kind of process.

Definition 1.2. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories. A functor from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is a family of functions* F *which associates an object* FA *in* $\mathbf{Y}$ *to each object* A *in* $\mathbf{X}$ *and a morphism* $Ff \in \mathrm{Hom}_{\mathbf{Y}}(FA, FB)$ *to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$*, and which is such that:*
(F1) $F(g \circ f) = Fg \circ Ff$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $F\,\mathrm{id}_A = \mathrm{id}_{FA}$ *for all* $A \in \mathrm{Ob}(\mathbf{X})$.

There is the definition of left and right adjoint functors. In the following we shall need two such adjoint constructions. First, in a given category the left adjoint of the diagonal functor (if it exists) is called the coproduct and the right adjoint (if it exists) is called the product: in **Sets**, the product is the Cartesian product and the coproduct is the disjoint union. Second, let the category $\mathbf{X}$ be concrete over some category $\mathbf{A}$ in the sense that there exists a faithful functor U from $\mathbf{X}$ to $\mathbf{A}$, usually called the forgetful functor. The left adjoint to this functor (if it exists) is then called the free functor. A standard example is the forgetful functor from complete metric spaces to metric spaces, whose left adjoint is the completion functor. On the next higher level of abstraction the notion of a *natural* transformation is settled. It is a kind of a function between functors and is defined as follows.

Definition 1.3. *Let* $F : \mathbf{X} \to \mathbf{Y}$ *and* $G : \mathbf{X} \to \mathbf{Y}$ *be two functors. A natural transformation* $\alpha : F \to G$ *is given by the following data.*

For every object A *in* $\mathbf{X}$*, there is a morphism* $\alpha_A : F(A) \to G(A)$ *in* $\mathbf{Y}$ *such that, for every morphism* $f : A \to B$ *in* $\mathbf{X}$ *the following diagram is commutative:*

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ F(f)\downarrow & & \downarrow G(f) \\ F(B) & \xrightarrow{\alpha_B} & G(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $G(f) \circ \alpha_A = \alpha_B \circ F(f)$.

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The morphisms α_A, $A \in \mathrm{Obj}(A)$, are called the components of the natural transformation α.

EXAMPLES

1.4. So, we can certainly speak, as before, of the "equality" of categories. We can also speak of the "isomorphism" of categories: an isomorphism between **C** and **D** is a functor $F : \mathbf{C} \to \mathbf{D}$ for which there is an inverse functor $G : \mathbf{D} \to \mathbf{C}$. I.e., FG is the identity functor on **C** and GF the identity on **D**, where we define the composition of functors in the obvious way. But because we also have natural transformations, we can also define a subtler notion, the "equivalence" of categories. An equivalence is a functor $F : \mathbf{C} \to \mathbf{D}$ together with a functor $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $a : FG \to 1_C$ and $b : GF \to 1_D$. A "natural isomorphism" is a natural transformation which has an inverse.

1.5. As we can "relax" the notion of equality to the notion of isomorphism when we pass from sets to categories, we can relax the condition that FG and GF equal identity functors to the condition that they be isomorphic to identity functor when we pass from categories to the 2-category **Cat**. We need to have the natural transformations to be able to speak of functors being isomorphic, just as we needed functions to be able to speak of sets being isomorphic. In fact, with each extra level in the theory of n-categories, we will be able to come up with a still more refined notion of "n-equivalence" in this way.

2. Method of Additional Structures on the Objects of a Category

2.1. Basic Definitions

To use the categorical language more efficient, we introduce the general concept of an additional structure on objects of a category [1]. This is the concept of concrete category but over any category.

In a category, two objects x and y can be equal or not equal, but they can be *isomorphic* or not, and if they are isomorphic, they can be isomorphic in many different ways. An isomorphism between x and y is simply a morphism $f : x \to y$ which has an inverse $g : y \to x$, such that $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$.

In the category **Sets**, an isomorphism is just a one-to-one and onto a function, i.e. a bijection. If we know two sets x and y are isomorphic we know that they are "the same in a way", even if they are not equal. But specifying an isomorphism $f : x \to y$ does more than say x and y are the same in a way; it specifies a *particular way* to regard x and y as the same.

In short, while equality is a yes-or-no matter, a mere *property*, an isomorphism is a *structure*. It is quite typical, as we climb the categorical latter (here from elements of a set to objects of a category) for properties to be reinterpreted as structures.

Definition 2.1. *We tell that a functor $F : \mathcal{C} \to \mathcal{C}'$ defines an additional $\mathcal{C}-$structure on objects of the category $\mathcal{C}'$ if*

1. *$\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, the* map $F : \mathcal{C}(X,Y) \to \mathcal{C}'(F(X), F(Y))$ *is injective,*
2. *$\forall X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{C}')$ and an isomorphism $u : Y \to F(X)$, there is an object $\tilde{Y} \in \mathrm{Ob}$ and an isomorphism $\tilde{u} : \tilde{Y} \to X$ such that $F(\tilde{Y}) = Y$ and $F(\tilde{u}) = u$.*

Such a functor is called a forgetful functor.

Almost all usual mathematical structures are structures on sets in this sense and there are corresponding forgetful functors to the category **Sets** of sets.

A forgetful functor $F : \mathcal{C} \to M(\mathcal{C}')$ defines a $\mathcal{C}$-structure *on morphisms of the category $\mathcal{C}'$*.

For our general structures, we can define usual constructions:

– inverse and direct images of structures,

– restrictions on subobjects,

– different products of structures.

We can define the category $\mathrm{Str}(\mathcal{C})$ of forgetful functors to the category $\mathcal{C}$. It is a full subcategory of the category $\mathrm{Cat}/\mathcal{C}$ of all categories over $\mathcal{C}$.

Let us consider some properties of structures (= forgetful functors):

– – In the category $\mathrm{Str}(\mathcal{C})$, the (bundle) product always exists. It gives a "union" structures.

– – Any functor $f : \mathcal{C} \to \mathcal{C}'$ transfers structures to the inverse direction, i.e., it defines the functor

$$f^* : \mathrm{Str}(\mathcal{C}') \to \mathrm{Str}(\mathcal{C}) : F \mapsto f^* F.$$

– – For a forgetful functor $F : \mathcal{C} \to \mathcal{C}'$, the functors

$$(F\circ) = \mathrm{Funct}(id, F) : \mathrm{Funct}(\mathcal{B}, \mathcal{C}) \to \mathrm{Funct}(\mathcal{B}, \mathcal{C}')$$

$$(\circ F) = \mathrm{Funct}(F, id) : \mathrm{Funct}(\mathcal{C}', \mathcal{B}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{B})$$

are forgetful functors.

– – One of constructions which transfers the structure
$F : \mathcal{C} \to$ Sets defined on sets to objects of any category $\mathcal{B}$, is the functor

$$h : \mathcal{B} \to \mathrm{Funct}(\mathcal{B}^\circ, \mathrm{Sets}) : B \mapsto h_B.$$

Thus, we have

$$\begin{array}{ccc} h_B^*\mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow F \\ \mathcal{B}' & \xrightarrow{h_B} & \mathrm{Sets} \end{array}$$

– – If a functor $A : \mathcal{B} \to \mathcal{C}$ is injective on morphisms (condition (1) in the definition of forgetful functor), then a forgetful functor $F : \mathcal{B}' \to \mathcal{C}$ and an equivalence $i : \mathcal{B} \to \mathcal{B}'$ exist, such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow_F & \\ \mathcal{B}' & & \end{array}$$

2.2. Structures on Topological Spaces

Among of structures on topological spaces we can select that, which is compatible with the topology. Let **Top** be a category of some topological spaces with a forgetful functor F : **Top** $\to$ **Sets**.

The categories associated with a topological space $T \in \mathrm{Ob(Top)}$ are as follows:

- The category $\mathcal{T}(T)$, where $\mathrm{Ob}(\mathcal{T}(T))$ is the set of all open subsets of T, and $\mathrm{Mor}(\mathcal{T}(T))$ is all their inclusions.
- The category (pseudogroup) $\mathcal{P}(T)$, where $\mathrm{Ob}(\mathcal{P}(T))$ is the set of all open subsets of T, and $\mathrm{Mor}(\mathcal{P}(T))$ is all their homeomorphisms.

Functors $\mathcal{T}(T)^\circ \to$ Sets are called presheaves of sets on T. Some of them are called sheaves. Thus we have the inclusions

$$\mathrm{Sh}(T) \subset \mathrm{Presh}(T) \subset \mathrm{Funct}(\mathcal{T}(T), \mathrm{Sets}).$$

The Grothendieck topology on a category is defined by saying which families of maps into an object constitute a covering of the object and certain axioms are fulfilled. A category together with the Grothendieck topology on it is called a *site*. For a site $\mathcal{C}$, one defines the full subcategory $\mathrm{Sh}(\mathcal{C}) \subset \mathrm{Presh}(\mathcal{C}) = \mathrm{Funct}(\mathcal{C}^\circ, \mathrm{Sets})$. The objects of $\mathrm{Funct}(\mathcal{C}^\circ, \mathrm{Sets})$ are called presheaves on the site $\mathcal{C}$, and the objects of $\mathrm{Sh}(\mathcal{C})$ are called sheaves on $\mathcal{C}$.

For any category there exists the finest topology such that the all representable presheaves are sheaves. It is called the canonical Grothendieck topology. Topos is a category which is equivalent to the category of sheaves for the canonical topology on them.

Hence, the topology is already transferred on a category so now, it is natural to consider all questions connected to local properties in the language of toposes and sheaves.

Here, we shall not consider local structures on toposes in general, and we shall restrict ourselves with the consideration of the elementary case of the category **Top**.

Definition 2.2. *A structure defined by a forgetful functor $f : \mathcal{C} \to$ Top is called a* **local structure** *if*

$\forall C \in \mathrm{Obj}(\mathcal{C})$ and any inclusion maps $i : U \to f(C)$ of the open subset U, an object $\tilde{U} \in Ob(\mathcal{C})$ and a morphism $\tilde{i} \in \mathcal{C}(\tilde{U}, C)$ exist such that $f(\tilde{U}) = U$, $f(\tilde{i}) = i$. This $\mathcal{C}-$structure $\tilde{U}$ is denoted by $C|U$ and called a **restriction** *of C on U.*

In other words, we can restrict ourselves with local structures on open subsets.

For a local structure $F : \mathcal{C} \to$ Top and each object $X \in \mathrm{Obj(Top)}$, there is the presheaf of categories

$$\mathcal{T}(X)^\circ \to \mathrm{Cat} : U \mapsto F^{-1}(U, \mathrm{id}_U).$$

Often this presheaf is a sheaf.

2.3. Structures on Smooth Manifolds

Let $\mathcal{M}$ be the category of smooth (∞-differentiable) manifolds with forgetful functor $f : \mathcal{M} \to$ Top, which defines a local structure and the presheaves of these structures are sheaves. On the category $\mathcal{M}$, there is the *tangent* functor $T : \mathcal{M} \to \mathcal{M} : M \mapsto T(M)$.

Its iterations give us almost all interesting functors on $\mathcal{M}$. Among them, we shall note the following:

- The cotangent functor $T^* : \mathcal{M} \to \mathcal{M} : M \mapsto T^*(M)$.
- For a manifold M and natural number $k = 0, 1, \ldots$, the functor of k-jets $J^k : \mathcal{M} \to \mathcal{M} : N \mapsto J^k(M, N)$.
- For a manifold M, $x \in M$, and natural number $k = 0, 1, \ldots$ the functor of k-jets at the point x $J_x^k : \mathcal{M} \to \mathcal{M} J_x^k(M, N)$.

Any category $\mathcal{C}$ of structures on smooth manifolds (or on $\mathcal{M}/$) has an additional structure, which gives us a possibility to define "smooth families of morphisms".

Definition 2.3. *Let $M, M', M'' \in \mathcal{M}$. A map*

$$\Phi : M \to \mathcal{M}(M', M'') : x \mapsto \Phi_x$$

is called a smooth family of morphisms if there exists a smooth map $\phi : M \times M' \to M''$ *such that*

$$\forall x \in M,\ x' \in M' \quad \Phi_x(x') = \phi(x, x').$$

Thus, we get the class of categories with smooth families and it appears the natural condition on functors.

Definition 2.4. *A functor is called a smooth functor if it maps each smooth family to a smooth family.*

Of course, all functors T, T^*, J^k, J^k_x are smooth.

2.4. Double Categories as an Additional Structure on Categories

In any category $\mathcal{C}$ with bundle products for some morphisms, we can define so-called intern categories. This is a monoid in the multiplicative category $\mathcal{C}//O$ of pairs of (special) morphisms $D, R : M \to O$ with the bundle product:

for $\xi = (D, R : M \to O)$ and $\xi' = (D', R' : M \to O)$ we get $\xi \star \xi' = (D \circ \pi_1, R' \circ \pi_2 : M \times_O M' \to O)$ where the unit objects $\mathrm{id}_M : 0 \to M$ and $\mathrm{id}_{M'} : 0 \to M'$, and the following diagram is commutative:

$$\begin{array}{ccc} M \times_O M' & \xrightarrow{\pi_2} & M' \\ {\scriptstyle\pi_1}\big\downarrow & & \big\downarrow{\scriptstyle R'} . \\ M & \xrightarrow{R} & O \end{array}$$

So an intern category is an object $\xi = (D, R : M- > O)$ with a multiplication $\mu : \xi \star \xi' \to \xi$ and the unit $\mathrm{id}_M : O \to M$.

Now we consider such an intern category as the category **Cat** of categories and will call it as *double categories* [2].

Definition 2.5. *A double category D consists of the following:*

(1) A category D_0 of objects $\mathrm{Obj}(D_0)$ *and morphisms* $\mathrm{Mor}(D_0)$ *of* 0*-level.*

(2) A category D_1 of morphisms $\mathrm{Obj}(D_1)$ *of* 1*-level and morphisms* $\mathrm{Mor}(D_1)$ *of* 2*-level.*

(3) Two functors $d, r : D_1 \rightrightarrows D_0$.

(4) A composition functor

$$* : D_1 \times_{D_0} D_1 \to D_1,$$

where the bundle product is defined by the commutative diagram

$$\begin{array}{ccc} D_1 \times_{D_0} D_1 & \xrightarrow{\pi_2} & D_1 \\ \pi_1 \downarrow & & \downarrow d \\ D_1 & \xrightarrow{r} & D_0 \end{array}$$

(5) A unit functor $ID : D_0 \to D_1$, which is a section of d, r.

There are strong and weak double categories.

Now we see that, for two objects $A, B \in \mathrm{Obj}(D_0)$ there are 0-level morphisms $D_0(A, B)$, which we note by ordinary arrows $f : A \to B$, and 1-level morphisms $D_{(1)}(A, B)$, which we note by the arrows $\xi : A \Rrightarrow B$ for $A = d(\xi)$ and $B = r(\xi)$. So with a 2-level morphism $\alpha : \xi \to \xi'$, where $\xi : A \Rrightarrow B$ and $\xi' : A' \Rrightarrow B'$, we can associate the diagram

$$\begin{array}{ccccc} A & \overset{\xi}{\Rrightarrow} & B & & \xi \\ d(\alpha) \downarrow & & \downarrow r(\alpha) & \longmapsto & \downarrow \alpha \\ A' & \overset{\xi'}{\Rrightarrow} & B' & & \xi' \end{array}$$

and arrow $\alpha : d(\alpha) \Rrightarrow r(\alpha)$.

On each level we have the corresponding compositions:

$$\begin{array}{cccc} \text{0-level} & \begin{array}{c} (A \xrightarrow{f} B \xrightarrow{g} C) \\ \xi \xrightarrow{\alpha} \eta \xrightarrow{\beta} \varsigma \end{array} & \begin{array}{c} \mapsto \\ \mapsto \end{array} & \begin{array}{c} g \circ f : A \to C \\ \beta \circ \alpha : \xi \to \varsigma \end{array} \\ \text{1-level} & (A \overset{\xi}{\Rrightarrow} B \overset{\eta}{\Rrightarrow} C) & \mapsto & \eta * \xi : A \Rrightarrow C \\ \text{2-level} & (f \overset{\alpha}{\Rrightarrow} g \overset{\beta}{\Rrightarrow} h) & \mapsto & \beta * \alpha : f \Rrightarrow h \end{array}$$

The composition on 2-level associated with the diagram

$$\begin{array}{ccccc} A & \overset{\xi}{\Rrightarrow} & B & & \xi \\ d(\alpha) \downarrow & & \downarrow r(\alpha) & & \downarrow \alpha \\ A' & \overset{\xi'}{\Rrightarrow} & B' & \longmapsto & \xi' \\ d(\alpha') \downarrow & & \downarrow r(\alpha') & & \downarrow \alpha' \\ A'' & \overset{\xi''}{\Rrightarrow} & B'' & & \xi'' \end{array}$$

Thus, a double category D consists of

- four sets $\mathrm{Obj}(D_0), \mathrm{Mor}(D_0), \mathrm{Obj}(D_1), \mathrm{Mor}(D_1)$, and eight maps of type d, r

$$\begin{array}{ccc} \mathrm{Obj}(D_1) & \leftleftarrows & \mathrm{Mor}(D_1) \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathrm{Obj}(D_0) & \leftleftarrows & \mathrm{Mor}(D_0) \end{array}$$

- two categories are associated D_0, D_1, and **almost categories**: $D_{(2)}$ with the set of objects $\mathrm{Obj}(D_0)$ and the set of morphisms $\mathrm{Obj}(D_1)$, $D_{(3)}$ with the set of objects $\mathrm{Mor}(D_0)$ and the set of morphisms $\mathrm{Mor}(D_1)$,

- $r, d : D_{(3)} \to D_{(2)}$ are almost functors.

Now we can define, for double categories, **double (category) functors** and their **morphisms**, **double subcategories**, the category $D\mathrm{Cat}$ of double categories, **equivalence** of double categories, **dual double categories** (changed direction of 1-level morphisms, i.e., d, r are transposed), and so on.

Definition 2.6. *A double category functor $F : D \to D'$ is a pair $F_0 : D_0 \to D'_0, F_1 : D_1 \to D'_1$ of usual functors such that*

$$\begin{aligned} & d' \circ F_1 = F_0 \circ d, \quad r' \circ F_1 = F_0 \circ r, \\ \forall\, \xi, \xi' \in \mathrm{Obj}(D_1) \quad & \varphi_{\xi,\xi'} : F_1(\xi * \xi') \xrightarrow{\sim} F_1(\xi) *' F_1(\xi'), \\ \forall\, A \in \mathrm{Obj}(D_0) \quad & \varphi_A : F_1(ID_A) \xrightarrow{\sim} ID_{F_0(A)}. \end{aligned}$$

2.5. Examples of Double Categories

Examples considered below show that double categories are sufficiently natural for mathematics.

Example 2.1. Bicategories are the partial case of a double category D when the category D_0 is trivial, i.e., it has only identical morphisms and compositions of 1-level and 2-level morphisms are associative.

Example 2.2. For each category C, we have the canonical double category $\mathrm{Morph}(C)$ of morphisms. Let C be a category, T be the diagram $\bullet \to \bullet$, TC be the category of diagrams in C of type T, let $D_0 = C$ and $D_1 = TC$. The functor d maps the diagram $f : A \to B$ into the object A, the functor r maps this diagram into the object B, and so on. It is easy to see that we get a double category D which is noted by $\mathrm{Morph}(C)$. Here $\mathrm{Obj}(D_1) = \mathrm{Mor}(D_0)$, a 2-level morphism $f \Rrightarrow g$ is a pair (u, v) of morphisms $u, v \in \mathrm{Mor}(C)$ with the usual composition from the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & A' \\ f\downarrow & & \downarrow f' \\ B & \xrightarrow{v} & B' \end{array}$$

Example 2.3. Let C be a category with bundle products, i.e., for all morphisms u, v to Y, the universal square

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow v \\ X & \xrightarrow{u} & Z \end{array}$$

exists. And let T be the following diagram

$\bullet \leftarrow \bullet \rightarrow \bullet$,

TC be the category of diagrams in C of type T. Now we define the double category D with $D_0 = D$ and $D_1 = TC$. Two functors

$d, r : TC \to C$,

where the functor d maps the diagram $A \leftarrow M \rightarrow B$ into the object A, the functor r maps this diagram into the object B. The composition: for two 1-level morphisms $\xi = (A \xleftarrow{\pi} M \xrightarrow{f} B) : A \Rrightarrow B$ and $\xi' = (B \xleftarrow{\pi'} M' \xrightarrow{f'} C) : B \Rrightarrow C$ we define their composition $\xi' \circ \xi = (A \xleftarrow{\pi\circ\pi_1} M \times_B M' \xrightarrow{f\circ\pi_2} C)$, where the bundle product is defined by the universal diagram

$$\begin{array}{ccc} M \times_B M' & \xrightarrow{\pi_2} & M' \\ \pi_1\downarrow & & \downarrow \pi' \\ M & \xrightarrow{f} & B \end{array}$$

A 2-level morphism is a triple $\alpha = (u, v, w) : \xi \to \xi'$ from the commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{f} & B & & \\ \pi\downarrow & \searrow v & & \searrow w & \\ A & & M' & \xrightarrow{f'} & B' \\ & \searrow u & \pi'\downarrow & & \\ & & A' & & \end{array}$$

with the evident composition.

Example 2.4. Let us consider a multiplicative (tensor) category $(C, \otimes, U, u)$. Then we have the double category with $D_1 = C$, and $D_0 = (*,*)$, e.c. a trivial category with one object and one morphism. The composition is

$$D_1 \times_{D_0} D_1 = C \times C \stackrel{\otimes}{\to} C.$$

Let us consider it in more details. Let $(C, \otimes, U, u)$ be a multiplicative (tensor) category with multiplication

$$\otimes : C \times C \to C : (X, Y) \mapsto X \otimes Y.$$

For the functor isomorphism of associativity

$$\varphi : \otimes \circ (\mathrm{id}, \otimes) \to \otimes \circ (\otimes, \mathrm{id}),$$

we write

$$\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z,$$

so the pentagon

$$\begin{array}{ccccc} X \otimes (Y \otimes (V \otimes W)) & \stackrel{\varphi_{X,Y,V\otimes W}}{\longrightarrow} & (X \otimes Y) \otimes (V \otimes W) & \stackrel{\varphi_{X\otimes Y,V,W}}{\longrightarrow} & ((X \otimes Y) \otimes V) \otimes W \\ \mathrm{id}_X \otimes \varphi_{Y,V,W} \downarrow & & & & \varphi_{X,Y,V} \otimes \mathrm{id}_W \uparrow \\ X \otimes ((Y \otimes V) \otimes W) & & \stackrel{\varphi_{X,Y\otimes V,W}}{\longrightarrow} & & (X \otimes (Y \otimes V)) \otimes W \end{array}$$

is commutative.

Then we have the double category D with $D_0 = C$ and D_1 such that

$$\mathrm{Obj}(D_1) = \{(X, x) | A, B, X \in \mathrm{Obj}(C), \quad x : X \otimes A \to B\}.$$

So, we write $\xi = (X, x) : A \Rrightarrow B$, and for $\xi \in \mathrm{Obj}(D_1)$, we denote $\xi = (X_\xi, x_\xi)$, $d(\xi) = A_\xi$, $\quad r(\xi) = B_\xi$. 2-level morphisms

$$D_1(\xi, \xi') = \{(f_1, f_2, f_3) \ \mid \textit{commutative diagram} \mid \ \begin{array}{ccc} X \otimes A & \stackrel{x}{\longrightarrow} & B \\ f_3 \otimes f_1 \downarrow & & \downarrow f_2' \\ X' \otimes A' & \stackrel{x'}{\longrightarrow} & B' \end{array} \ \}$$

and $d(f_1, f_2, f_3) = f_1, \quad r(f_1, f_2, f_3) = f_2$.

Composition $D_1 \times_{D_0} D_1 \to D_1$ is defined as follows

for $A \stackrel{\xi}{\Rrightarrow} B \stackrel{\xi'}{\Rrightarrow} B' \quad \xi \circ \xi' = (A, B', X'X, x'')$, where x'' is the following composition

$$(X' \otimes X) \otimes A \stackrel{\varphi^{-1}_{X',X,A}}{\longrightarrow} X' \otimes (X \otimes A) \stackrel{\mathrm{id}_{X'} \otimes x}{\longrightarrow} X' \otimes B \stackrel{x'}{\longrightarrow} B'.$$

Associativity. For $A \stackrel{\xi}{\Rrightarrow} B \stackrel{\xi'}{\Rrightarrow} B' \stackrel{\xi''}{\Rrightarrow} B''$, the left column gives $x_{\xi''\circ(\xi'\circ\xi)}$, the right column gives $x_{(\xi''\circ\xi')\circ\xi}$

$$\begin{array}{ccc} (X'' \otimes (X' \otimes X)) \otimes A & & ((X'' \otimes X') \otimes X) \otimes A \\ \varphi^{-1}_{X'',X'\otimes X,A} \downarrow & & \varphi^{-1}_{X''\otimes X',X,A} \downarrow \\ X'' \otimes ((X' \otimes X) \otimes A) & & (X'' \otimes X') \otimes (X \otimes A) \\ \mathrm{id}_{X''} \otimes \varphi^{-1}_{X',X,A} \downarrow & & \mathrm{id}_{X''\otimes X'} \otimes x \downarrow \\ X'' \otimes (X' \otimes (X \otimes A)) & & (X'' \otimes X') \otimes B \\ \mathrm{id}_{X''} \otimes (\mathrm{id}_{X'} \otimes x) \downarrow & & \varphi^{-1}_{X'',X',B} \downarrow \\ X'' \otimes (X' \otimes B) & = & X'' \otimes (X' \otimes B) \\ \mathrm{id}_{X''} \otimes x' \downarrow & & \mathrm{id}_{X''} \otimes x' \downarrow \\ X'' \otimes B'' & = & X'' \otimes B' \\ x'' \downarrow & & x'' \downarrow \\ B'' & & B'' \end{array}$$

So we have isomorphism

$$(\varphi_{X'',X',X}, \mathrm{id}_{A'}, \mathrm{id}_{B'}) : \xi'' \circ (\xi' \circ \xi) \to (\xi'' \circ \xi') \circ \xi.$$

2.5.1. Bundle of Categories

Let $\varphi : \mathcal{F} \to \mathcal{C}$ be a functor and $U \in \mathrm{Obj}\mathcal{C}$ for all objects. We denote by $\mathcal{F}_U = \varphi^{-1}(U, \mathrm{id}_U)$ the subcategory of F with

$$\begin{aligned} \mathrm{Obj}\,\mathcal{F}_U &= \{u \in \mathrm{Obj}\mathcal{F} \mid \varphi(u) = U\}, \\ \mathrm{Mor}\,\mathcal{F}_U &= \{f \in \mathrm{Mor}\,\mathcal{F} \mid \varphi(f) = \mathrm{id}_U\}. \end{aligned}$$

Let $(f : v \to u) \in \mathrm{Mor}\mathcal{F}$, $\varphi(f : v \to u) = (g : V \to U)$. Then one tells that f is Descartes's morphism, or that v is inverse image $g^*(u)$ of the object u, if $\forall v' \in \mathrm{Obj}(\mathcal{F}_V)$ and the map

$$f_* : \mathcal{F}_V(v', v) \to \mathcal{F}_g(v', u) : h \mapsto f \circ h$$

is a bijection. Here, we have

$$\mathcal{F}_g(v, u) \stackrel{\mathrm{def}}{=} \{h \in \mathcal{F}(v, u) \mid \varphi(h) = g\}.$$

So we have the diagram

$$\begin{array}{ccc} \forall\, v' & & \\ \downarrow^{h} & \searrow^{f\circ h} & \\ v & \underset{f}{\longrightarrow} & u \\ & & \\ V & \overset{g}{\longrightarrow} & U \end{array}$$

A functor $P : \mathcal{F} \to \mathcal{C}$ is called a bundle of categories if inverse images allows exist and a composition of two Descartes morphisms is a Descartes morphism too. Then g^* may be transferred to a functor $\mathcal{F}(U) \to \mathcal{F}(V)$, and $(g_1 \circ g_2)^*$ will be canonical isomorphic to $g_2^* \circ g_1^*$.

Example 2.5. The projection

$$\Pi_1 : \mathrm{Mor(Top)} \to \mathrm{Top} : (f : X \to Y) \mapsto X$$

is a bundle of categories. For different structures on topological spaces, it is not always truth for the category of all morphisms, but may be truth for a subcategory.

Example 2.6. Let **Sub** be a subcategory in Mor(Man) consisting from submersions. Then the projection

$$\Pi_2 : \mathbf{Sub} \to \mathrm{Man} : (f : X \to Y) \mapsto Y$$

is a bundle of categories and for each morphism $h \in \mathrm{Man}(B', B)$, we have the functor of **inverse image**:

$$h^* : \mathbf{Sub}_B \to \mathbf{Sub}_{B'} : (f : M \to B) \mapsto (B' \times_B M \to B').$$

The set $\Gamma(\pi)$ of sections of an submersion $\pi : M \to B$ is the set of morphisms $\mathbf{Sub}(\mathrm{id}_B, \pi)$.

Example 2.7. Let **Mod** be the category of pairs (R, M), where R is a ring and M is a left R-module. Let **Rings** be the category of rings. Then the functor

$$\mathrm{Mod} \to \mathrm{Rings} : (R, M) \mapsto R$$

is a bundle of categories and for each morphism $h \in \mathbf{Ring}(R', R)$, we have the functor of **inverse image**:

$$h^* : R\text{-}\mathbf{mod} \to R'\text{-}\mathbf{mod} : M \mapsto R' \otimes_R M.$$

2.6. Fibers of Functor Morphisms

The Grothendieck's definition of a fiber of a functor morphism is applicable to morphisms of functors from any category to the category **Sets** of sets. Let $F, G : \mathcal{C} \to \mathrm{Set}$, and $\varphi : F \to G$ be their morphism. For each object $S \in \mathrm{Obj}(\mathcal{C})$ and an element $\alpha \in G(S)$, the fiber φ_α of φ over α is the functor

$$\varphi_\alpha : \mathcal{C}/S \to \mathrm{Sets} : f \mapsto \varphi_\alpha(f),$$

where, for a morphism $f : T \to S$,

$$\varphi_\alpha(f) = \{\beta \in F(T) \mid G(f) \circ \varphi_T(\beta) = \alpha\}.$$

So we have the diagram

$$\begin{array}{ccccc} \varphi^\alpha(f) \subset & F(T) & & F(S) & \\ & \varphi_T \downarrow & & \downarrow \varphi_S & \\ & G(T) & \overset{G(f)}{\longrightarrow} & G(S) & \ni \alpha. \end{array}$$

3. Multiplicative Structures on Categories

3.1. Concepts and State of the Art

The prototype of a category is the category **Sets** of sets and functions. The prototype of a 2-category is the category **Cat** of small categories and functors. **Cat** has a more structure on it than a simple category because we have natural transformations between functors. This can be viewed in the following way: The extra structure implies that every morphism set $\mathrm{Hom}(C, D)$ in **Cat** is actually not only a set but a category itself, where the composition and identities in **Cat** are compatible with this categorical structure on the Hom-sets (i.e., composition and identities are functorial with respect to the structure on the Hom-sets). A general category with this kind of extra structure is called a 2-category.

The definition of 2-category can be put in a more general setting (which will be convenient below) by using the language of enriched categories. A category $\mathcal{C}$ is enriched over a category $\mathcal{V}$ if every Hom-set in $\mathcal{C}$ has the structure of an object in $\mathcal{V}$ and if the composition and identities in $\mathcal{C}$ are compatible with this extra structure on the Hom-sets. So, a 2-category is a category enriched over **Cat**. Now, the (small) 2-categories again form a category **2-Cat** and a 3-category can be defined as a category enriched over **2-Cat** (indeed, **2-Cat** turns out to be a 3-category itself). In this way we can proceed iteratively to define n-categories and then ω-categories as categories involving n-categorical structures of all levels.

A specific recipe for obtaining monoidal (braided, etc.) 2-categories via Hopf categories is proposed by Crane and Frenkel [3]. Namely, that it is supposed the 2-category of module-categories over a Hopf category now plays an important role in 4-dimensional topology and TQFT. Although the theory of Hopf categories is devised, in general, by Neuchl [4], interesting examples are still missing. In particular, the Hopf category, underlying the Lusztig's canonical basis [5] of a quantized universal enveloping algebra, is not constructed yet. It was proposed to define it as a family of abelian categories of perverse l-adic sheaves equipped with some functors of multiplication and comultiplication [6]. These perverse sheaves are equivariant in the sense of Bernstein and Lunts [7].

It turns out that the notions of n-category and ω-category are not general enough for several interesting applications. What one gets there are weak versions of these concepts (instead of weak n-category, sometimes the notions of bicategory, tricategory, etc. are used). Let us shortly explain what this means: In a category, it does not make sense to ask for equality of objects but the appropriate notion is isomorphism. In the same way, in a 2-category, we should not ask for equality of morphisms but only for equality up to an invertible 2-morphism (the morphisms between the morphisms, e. g., the natural transformations in **Cat**). Applying this to the categorical structure itself (i.e., requiring associativity and identity properties only up to natural equivalence) leads to the notion of weak 2-category (or bicategory). In the same way, we can weaken the structure of an n-category up to the $(n-1)$-th level to obtain a weak n-category.

The point making this weakening to be an involved matter is that, in general, we need so-called coherence conditions in addition to the weakened laws in order to assure that some properties, known from the strict case, hold. E.g., to assure that associativity is iteratively applicable (i.e., that we can up to a 2-isomorphism rebracket composites involving more than three factors), we need a coherence condition stating that even four factors can be rebracketed (and the other cases follow then). See the literature given above for details.

No satisfactory versions of a weak n-category for higher n and of a weak ω-category were available for a long time but now there are several approaches at hand [8–10]. The relationship between these approaches and a universal understanding of these structures have still to be achieved.

3.2. Multiplicative Categories

Definition 3.1. *A* **multiplication** *in the category $\mathcal{C}$ is an associative functor*

$$*:\mathcal{C}\times\mathcal{C}\to\mathcal{C}:(X,Y)\mapsto X*Y.$$

An **associativity morphism** *for $*$ is a functor isomorphism*

$$\varphi_{X,Y,Z}:X*(Y*Z)\to(X*Y)*Z$$

such that, for any four objects X,Y,Z,T the following diagram is commutative:

$$X*(Y*(Z*T)) \quad \stackrel{\varphi_{X,Y,Z*T}}{\longrightarrow} \quad (X*Y)*(Z*T) \quad \stackrel{\varphi_{X*Y,Z,T}}{\longrightarrow} \quad X*Y*Z*T$$
$$\downarrow \mathrm{id}_X*\varphi_{Y,Z,T} \qquad\qquad\qquad\qquad \uparrow \varphi_{X,Y,Z}*\mathrm{id}_T$$
$$X*((Y*Z)*T) \qquad \stackrel{\varphi_{X,Y*Z,T}}{\longrightarrow} \qquad (X*(Y*Z))*T$$

$$\begin{tikzcd}
X*(Y*(Z*T)) \arrow[r, "\varphi_{X,Y,Z*T}"] \arrow[d, "\mathrm{id}_X*\varphi_{Y,Z,T}"'] & (X*Y)*(Z*T) \arrow[r, "\varphi_{X*Y,Z,T}"] & X*Y*Z*T \\
X*((Y*Z)*T) \arrow[rr, "\varphi_{X,Y*Z,T}"] & & (X*(Y*Z))*T \arrow[u, "\varphi_{X,Y,Z}*\mathrm{id}_T"']
\end{tikzcd}$$

A **commutativity morphism** *for* $*$ *is a functor isomorphism*

$$\psi_{X,Y} : X * Y \to Y * X$$

such that, for any two objects X, Y*, we have*

$$\varphi_{X,Y} \circ \varphi_{Y,X} = \mathrm{id}_{X*Y} : X * Y \to X * Y.$$

The associativity φ *of morphisms and commutativity* ψ *are compatible if, for any three objects* X, Y, Z*, the following diagram is commutative:*

$$\begin{array}{ccccc} X * (Y * Z) & \overset{\varphi_{X,Y,Z}}{\longrightarrow} & (X * Y) * Z & \overset{\psi_{X*Y,Z}}{\longrightarrow} & Z * (X * Y) \\ \downarrow \mathrm{id}_X * \psi_{Y,Z} & & & & \uparrow \varphi_{Z,X,Y} \\ X * (Z * Y) & \overset{\varphi_{X,Z,Y}}{\longrightarrow} & (X * Z) * Y & \overset{\psi_{X,Z}*\mathrm{id}_Y}{\longrightarrow} & (Z * X) * Y \end{array}$$

$$\begin{array}{ccccc} X * (Y * Z) & \overset{\varphi_{X,Y,Z}}{\longrightarrow} & (X * Y) * Z & \overset{\psi_{X*Y,Z}}{\longrightarrow} & Z * (X * Y) \\ \downarrow \mathrm{id}_X * \psi_{Y,Z} & & & & \uparrow \varphi_{Z,X,Y} \\ X * (Z * Y) & \overset{\varphi_{X,Z,Y}}{\longrightarrow} & (X * Z) * Y & \overset{\psi_{X,Z}*\mathrm{id}_Y}{\longrightarrow} & (Z * X) * Y \end{array}$$

A pair (U, u) *where* $U \in \mathrm{Obj}(\mathcal{C})$ *and an isomorphism* $u : U \to U * U$ *is called a* **unit object** *for* $\mathcal{C}, *$ *if the functor*

$$X \mapsto U * X : \mathcal{C} \to \mathcal{C}$$

is the equivalence of categories.

Definition 3.2. *A* **multiplicative category** *is a collection* $(\mathcal{C}, *, \varphi, \psi, U, u)$.

If there are some additional structures on category, then it is usually assumed that product $*$ and other elements of the collection are compatible with these structures.

3.3. $\mathcal{C}$-monoids or Multiplicative Objects. Monoidal Categories and Monoids. Comonoids

Let $\mathbf{C} = (\mathcal{C}, *, \varphi, \psi, U, u)$ be a multiplicative category. A **multiplicative object** in $\mathbf{C}$ or $\mathcal{C}$**-monoid** is an object $M \in \mathrm{Obj}(\mathcal{C})$ with multiplication $\mu : M * M \to M : (m, m') \mapsto \mu(m, m')$ and a unit $\varepsilon : U \to M$ such that the following axioms are faithful:

(1) Associativity: the following diagram is commutative:

$$\begin{array}{ccccc} M * (M * M) & & \overset{\varphi_{M,M,M}}{\longrightarrow} & & (M * M) * M \\ \downarrow \mathrm{id}_M * \mu & & & & \downarrow \mu * \mathrm{id}_M \\ M * M & \overset{\mu}{\longrightarrow} & M & \overset{\mu}{\longleftarrow} & M * M \end{array}$$

(2) Unit: the following diagram is commutative:

$$\begin{array}{ccccc} M & \to & U * M & \overset{\psi_{U*M}}{\to} & M * U \\ || & & \downarrow \varepsilon * \mathrm{id}_M & & \downarrow \mathrm{id}_M * \varepsilon \\ M & \overset{\mu}{\to} & M * M & = & M * M \end{array}$$

Examples 3.1. Let R be a commutative ring. The category R**-mod** of R-modules is a multiplicative category under the tensor product $\otimes_R$, where the unit object is the left R-module R. Multiplicative objects in the category are R-algebras with units.

Examples 3.2. A small multiplicative category $\mathcal{C}$ is a multiplicative object of the multiplicative category Sets$//\mathrm{Obj}(\mathcal{C})$.

Multiplicative structures may be described in categories as monoids in a monoidal category.

A monoidal category $(\mathcal{C}, \otimes, K, \varphi, \dots)$ consists of:
$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $K \in \mathrm{Ob}\mathcal{C}$ – the unit object,
and the functor-isomorphisms:

$$\varphi_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$\psi_A : A \otimes K \to A, \ldots$, where $\otimes$ is symmetric, if there exists a functor-isomorphism

$$\theta_{A,B} : A \otimes B \to B \otimes A.$$

A monoid in a monoidal category $(\mathcal{C}, \otimes, K, \varphi, \ldots)$ is an object M endowed by a multiplication

$$\mu : M \otimes M \to M$$

and the unit morphism $\varepsilon : K \to M$+Axioms.

A comonoid is a monoid in $(\mathcal{C}^{op}, \otimes, K, \varphi, \ldots)$. In $\mathcal{C}$, we have the comultiplication

$$\Delta : M \to M \otimes M$$

the counit $\eta : M \to K$+Axioms.

An action of a monoid M on A is defined by

$$\alpha : M \otimes A \to A$$

+ Axioms.

A monoidal functor (a morphism of monoidal categories) of two monoidal categories is defined as $F : (\mathcal{C}, \otimes, K) \to (\mathcal{C}', \otimes', K')$ if

$$F(A \otimes B) \approx F(A) \otimes' F(B)$$

and $F(K) \approx K'$.

Example 3.3. A monoidal category is a monoid in the monoidal category $(\mathrm{Cat}, \times, K)$ of categories with Cartesian product. Here, K is a category with only one object $*$ and only one morphism id_*.

Example 3.4. The category Symm with objects $[n]$ for $n = 0, 1, \ldots$ and morphisms

$$\mathrm{Symm}([n],[m]) = \begin{cases} \emptyset, & \text{if } n \neq m, \\ \Sigma_n, & \text{if } n = m, \end{cases}$$

where Σ_n is the group of permutations of $(1, \ldots, n)$, with the multiplication

$$* : \mathrm{Symm} \times \mathrm{Symm} \to \mathrm{Symm}$$

such that $[n] * [m] \approx [n + m - 1]$ with the following identification of the inputs:

$$(1, \ldots, n) * (\overline{1}, \ldots, \overline{m}) = (1, \ldots, n, \overline{2}, \ldots, \overline{m})$$

which explanes the action of $*$ on morphisms.

Example 3.5. Let $(\mathcal{C}, \otimes, K)$ and $(\mathcal{C}', \otimes', K')$ be two monoidal catigories, $F \in \mathrm{Ob}(\mathcal{C}\mathcal{C}')$ and $F(K) = K'$. Then on the category of such functors F, there is a monoidal structure, and a monoid is defined by a functor morphism

$$\mu_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$$

with the natural axioms of associativity and unit.

Example 3.6. bialgebras and Dual construction

Algebras as monoids in k-bf vect, k-**alg**, Bialgebras as comonoids in k-**alg**, k-**bialg**; double Categories as monoids in the category of pairs of functors.

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МЕТОД ДОДАТКОВИХ СТРУКТУР
НА ОБЄ'КТАХ КАТЕГОРІЙ ЯК ОСНОВА
ТЕОРЕТИКО-КАТЕГОРІЙНОГО АНАЛІЗУ В ФІЗИЦІ

С. Москалюк

Р е з ю м е

Ми переходимо від окремого об'єкта до групи його автоморфізмів і до і до інваріантів цієї групи, потім до категорії таких об'єктів і функторів на категорії. Таким чином на зміну теорії інваріантів приходить теорія еквіваріантних відображень й тоді виникає проблема як описати функтори із одної категорії в іншу. Як писав Герман Вейль: “Фелікс Клейн зрозумів і застосував поняття групи як універсальний структуруючий принцип в алгебрі, геометрії та аналізі”. Сьогодні ця місія груп переходить до категорій.

МЕТОД ДОПОЛНИТЕЛЬНИХ СТРУКТУР
НА ОБЪЕКТАХ КАТЕГОРИЙ КАК ОСНОВА
ТЕОРЕТИКО-КАТЕГОРИЙНОГО АНАЛИЗА В ФИЗИКЕ

С. Москалюк

Р е з ю м е

Мы переходим от отдельного объекта к группе его автоморфизмов и к инвариантам этой группы, затем к категории таких объектов и функторам на категории. Таким образом на смену теории инвариантов приходит теория эквивариантных отображений и тогда возникает проблема описания функторов из одной категории в другую. Как отметил Герман Вейль: “Феликс Клейн осознал и применил понятие группы как универсальний структурирующий принцип в алгебре, геометрии и анализе”. В настоящее время эта миссия групп переходит к категориям.