REPRESENTATIONS OF THE ALGEBRA $U'_q(\mathrm{so}_n)$ RELATED TO QUANTUM GRAVITY[1]

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The aim of this paper is to review our results on finite dimensional irreducible representations of the nonstandard q-deformation $U'_q(\mathrm{so}_n)$ of the universal enveloping algebra $U(\mathrm{so}(n))$ of the Lie algebra $\mathrm{so}(n)$ which does not coincide with the Drinfeld–Jimbo quantum algebra $U_q(\mathrm{so}_n)$. This algebra is related to algebras of observables in quantum gravity and to algebraic geometry. Irreducible finite dimensional representations of the algebra $U'_q(\mathrm{so}_n)$ for q not a root of unity and for q a root of unity are given.

1. Introduction

Quantum orthogonal groups, quantum Lorentz groups, and their corresponding quantized universal enveloping algebras are of special interest for modern mathematics and physics. M. Jimbo [1] and V. Drinfeld [2] defined q-deformations (quantized universal enveloping algebras) $U_q(g)$ for all simple complex Lie algebras g by means of Cartan subalgebras and root subspaces (see also [3] and [4]). However, these approaches do not give a satisfactory presentation of the quantized algebra $U_q(\mathrm{so}(n))$ from a viewpoint of some problems in physics and mathematics. When considering finite dimensional representations of the quantum groups $\mathrm{SO}_q(n+1)$ and $\mathrm{SO}_q(n,1)$, we are often interested in reducing them onto the quantum subgroup $\mathrm{SO}_q(n)$. This reduction would give an analogue of the Gel'fand–Tsetlin basis for those representations which are often used in physics. However, the definitions of the quantized universal enveloping algebras mentioned above do not allow the inclusions $U_q(\mathrm{so}(n+1)) \supset U_q(\mathrm{so}(n))$ and $U_q(\mathrm{so}(n,1)) \supset U_q(\mathrm{so}(n))$. To be able to obtain such reductions, we have to consider q-deformations of the universal enveloping algebra $U(\mathrm{so}(n))$, when $\mathrm{so}(n)$ is defined in terms of the generators $I_{k,k-1} = E_{k,k-1} - E_{k-1,k}$ (where E_{is} is the matrix with elements $(E_{is})_{rt} = \delta_{ir}\delta_{st}$) rather than by means of the Cartan subalgebra and root elements. To construct such deformations, we have to deform trilinear relations for elements $I_{k,k-1}$ instead of Serre's relations (used in the case of the quantized universal enveloping algebras of Drinfeld and Jimbo). As a result, we obtain the associative algebra which will be denoted as $U'_q(\mathrm{so}_n)$.

This q-deformation was first constructed in [5]. It allows us to construct the reductions of $U'_q(\mathrm{so}_{n,1})$ and $U'_q(\mathrm{so}_{n+1})$ onto $U'_q(\mathrm{so}_n)$. The q-deformed algebra $U'_q(\mathrm{so}_n)$ leads for $n=3$ to the q-deformed algebra $U'_q(\mathrm{so}_3)$ defined by D. Fairlie [6]. The cyclically symmetric algebra, similar to Fairlie's one, was also considered somewhat earlier by Odesskii [7]. The algebra $U'_q(\mathrm{so}_4)$ is a q-deformation of the algebra $U(\mathrm{so}(4))$. In the case of the classical Lie algebra $\mathrm{so}(4)$ one has $\mathrm{so}(4) = \mathrm{so}(3) \oplus \mathrm{so}(3)$, while it is not a case for our q-deformation $U'_q(\mathrm{so}_4)$ (see [8]).

In the classical case, the imbedding $\mathrm{SO}(n) \subset \mathrm{SU}(n)$ (and its infinitesimal analogue) is of great importance for nuclear physics and in the theory of Riemannian symmetric spaces. It is well known that, in the framework of Drinfeld–Jimbo quantized universal enveloping algebras and the corresponding quantum groups, one cannot construct such embedding. The algebra $U'_q(\mathrm{so}_n)$ allows one to define such an embedding [9], that is, it is possible to define the embedding $U'_q(\mathrm{so}_n) \subset U_q(\mathrm{sl}_n)$, where $U_q(\mathrm{sl}_n)$ is the Drinfeld–Jimbo quantized enveloping algebra.

As a disadvantage of the algebra $U'_q(\mathrm{so}_n)$ we have to mention the difficulties with Hopf algebra structure. Nevertheless, $U'_q(\mathrm{so}_n)$ turns out to be a coideal in $U_q(\mathrm{sl}_n)$ (see [9]) and this fact allows us to consider tensor products of finite dimensional irreducible representations of $U'_q(\mathrm{so}_n)$ in many interesting cases (see [10]).

Finite dimensional irreducible representations of the algebra $U'_q(\mathrm{so}_n)$ were constructed in [5]. The formulas of action of the generators of $U'_q(\mathrm{so}_n)$ upon the basis (which is a q-analogue of the Gel'fand–Tsetlin basis) are given there. A proof of these formulas and some their corrections were given in [11].

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Finite dimensional irreducible representations described in [5] and [11] are representations of the classical type. They are q-deformations of the corresponding irreducible representations of the Lie algebra so(n), that is, they turn into representations of so(n) at $q \to 1$.

The algebra $U'_q(\mathrm{so}_n)$ has other classes of finite dimensional irreducible representations which have no classical analogue. These representations are singular at the point $q = 1$. They are described in [12]. A description of these representations for the algebra $U'_q(\mathrm{so}_3)$ is given in [13]. A classification of irreducible $*$-representations of real forms of the algebra $U'_q(\mathrm{so}_3)$ is given in [14].

Irreducible representations of $U'_q(\mathrm{so}_n)$, when q is a root of unity were considered in [15]. In this case, all irreducible representations of $U'_q(\mathrm{so}_n)$ are finite dimensional. In order to prove the corresponding theorem, an analogue of the Poincaré–Birkhoff–Witt theorem for $U'_q(\mathrm{so}_n)$ and the description of central elements of this algebra for q a root of unity (given in [16]) were used. For construction of irreducible representations of $U'_q(\mathrm{so}_n)$ for q a root of unity, the method of D. Arnaudon and A. Chakrabarti [17] for construction of irreducible representations of the quantum algebra $U_q(\mathrm{sl}_n)$ when q is a root of unity was applied.

The aim of this paper is to give a review of the results on finite dimensional irreducible representations of the algebra $U'_q(\mathrm{so}_n)$ obtained in Bogolyubov Institute for Theoretical Physics (Kiev).

The algebra $U'_q(\mathrm{so}_n)$ is interesting because of its applications in quantum gravity [18–20] and in the theory of harmonics on a quantum vector space (see [21] and [22]).

2. The q-deformed Algebra $U'_q(\mathrm{so}_n)$

The origin of existing a q-deformation of the universal enveloping algebra $U(\mathrm{so}(n))$, different from the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathrm{so}_n)$, consists in the following. The Lie algebra so(n) has two structures:

(a) The structure related to existing a Cartan subalgebra and root elements in so(n). A quantization of this structure leads to the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathrm{so}_n)$.

(b) The structure related to realization of so(n) by skew-symmetric matrices. In so(n), there exists a basis consisting of the matrices I_{ij}, $i > j$, defined as $I_{ij} = E_{ij} - E_{ji}$, where E_{ij} is the matrix with entries $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$. These matrices are not root elements.

Using structure (b), we may say that the universal enveloping algebra $U(\mathrm{so}(n))$ is generated by the elements I_{ij}, $i > j$. But in order to generate the universal enveloping algebra $U(\mathrm{so}(n))$, it is enough to take only the elements $I_{21}, I_{32}, \cdots, I_{n,n-1}$. It is a minimal set of elements necessary for generating the algebra $U(\mathrm{so}(n))$. These elements satisfy the relations

$$I^2_{i,i-1}I_{i+1,i} - 2I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I^2_{i,i-1} = -I_{i+1,i},$$

$$I_{i,i-1}I^2_{i+1,i} - 2I_{i+1,i}I_{i,i-1}I_{i+1,i} + I^2_{i+1,i}I_{i,i-1} = -I_{i,i-1},$$

$$I_{i,i-1}I_{j,j-1} - I_{j,j-1}I_{i,i-1} = 0 \quad \text{for} \quad |i-j| > 1.$$

The following theorem is true [23] for the algebra $U(\mathrm{so}(n))$: *The universal enveloping algebra $U(\mathrm{so}(n))$ is isomorphic to the complex associative algebra (with a unit element) generated by the elements I_{21}, $I_{32}, \cdots, I_{n,n-1}$ satisfying the above relations.*

We make the q-deformation of these relations by deforming the integer 2 in these relations as

$$2 \to [2] := (q^2 - q^{-2})/(q - q^{-1}) = q + q^{-1}.$$

As a result, we obtain the complex associative algebra (with a unit element) generated by the elements I_{21}, $I_{32}, \cdots, I_{n,n-1}$ satisfying the relations

$$I^2_{i,i-1}I_{i+1,i} - (q+q^{-1})I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I^2_{i,i-1} = = -I_{i+1,i}, \tag{1}$$

$$I_{i,i-1}I^2_{i+1,i} - (q+q^{-1})I_{i+1,i}I_{i,i-1}I_{i+1,i} + I^2_{i+1,i}I_{i,i-1} = = -I_{i,i-1}, \tag{2}$$

$$I_{i,i-1}I_{j,j-1} - I_{j,j-1}I_{i,i-1} = 0 \quad \text{for} \quad |i-j| > 1. \tag{3}$$

This algebra was introduced by us in [5] and is denoted by $U'_q(\mathrm{so}_n)$.

The analogue of the elements $I_{ij}, i > j$, can be introduced into $U'_q(\mathrm{so}_n)$ (see [24] and [25]). In order to give them, we use the notation $I_{k,k-1} \equiv I^+_{k,k-1} \equiv I^-_{k,k-1}$. Then, for $k > l+1$, we define recursively

$$I^+_{kl} := [I_{l+1,l}, I_{k,l+1}]_q \equiv \equiv q^{1/2}I_{l+1,l}I_{k,l+1} - q^{-1/2}I_{k,l+1}I_{l+1,l}, \tag{4}$$

$$I^-_{kl} := [I_{l+1,l}, I_{k,l+1}]_{q^{-1}} = q^{-1/2}I_{l+1,l}I_{k,l+1} - q^{1/2}I_{k,l+1}I_{l+1,l}$$

The elements I^+_{kl}, $k > l$, satisfy the commutation relations

$$[I^+_{ln}, I^+_{kl}]_q = I^+_{kn}, \quad [I^+_{kl}, I^+_{kn}]_q = I^+_{ln}, \quad [I^+_{kn}, I^+_{ln}]_q = I^+_{kl} \tag{5}$$

for $\quad k>l>n$,

$$[I_{kl}^+, I_{nr}^+]=0 \quad \text{for} \quad k>l>n>r \;\text{ and }\; k>n>r>l, \tag{6}$$

$$[I_{kl}^+, I_{nr}^+]_q=\lambda(I_{lr}^+ I_{kn}^+ - I_{kr}^+ I_{nl}^+) \quad \text{for} \quad k>n>l>r. \tag{7}$$

where $\lambda = q-q^{-1}$. For I_{kl}^-, $k>l$, the commutation relations are obtained from these relations by replacing I_{kl}^+ by I_{kl}^- and q by q^{-1}.

The algebra $U_q'(\mathrm{so}_n)$ can be defined as a unital associative algebra generated by I_{kl}^+, $1\le l<k\le n$, satisfying relations (5)–(7). In fact, using relations (4) we can reduce relations (5)–(7) to relations (1)–(3) for $I_{21}, I_{32}, \cdots, I_{n,n-1}$.

The Poincaré–Birkhoff–Witt theorem for the algebra $U_q'(\mathrm{so}_n)$ can be formulated as follows (a proof of this theorem is given in [15]): *The elements*

$$I_{21}^{+\;m_{21}} I_{31}^{+\;m_{31}} \cdots I_{n1}^{+\;m_{n1}} I_{32}^{+\;m_{32}} I_{42}^{+\;m_{42}} \cdots I_{n2}^{+\;m_{n2}} \cdots \times$$

$$\times I_{n,n-1}^{+\;m_{n,n-1}}, \qquad m_{ij}=0,1,2,\cdots, \tag{8}$$

form a basis of the algebra $U_q'(\mathrm{so}_n)$. This assertion is true if I_{ij}^+ are replaced by the corresponding elements I_{ij}^-.

Example 1. Let us consider the case of the algebra $U_q'(\mathrm{so}_3)$. It is generated by two elements I_{21} and I_{32} satisfying the relations

$$I_{21}^2 I_{32} - (q-q^{-1}) I_{21} I_{32} I_{21} + I_{32} I_{21}^2 = -I_{32}, \tag{9}$$

$$I_{21} I_{32}^2 - (q+q^{-1}) I_{32} I_{21} I_{32} + I_{32}^2 I_{21} = -I_{21}. \tag{10}$$

Introducing the element $I_{31}=q^{1/2} I_{21} I_{32} - q^{-1/2} I_{32} I_{21}$, we have the relations

$$[I_{21}, I_{32}]_q = I_{31},\ [I_{32}, I_{31}]_q = I_{21},\ [I_{31}, I_{21}]_q = I_{32}, \tag{11}$$

where the q-commutator $[\cdot,\cdot]_q$ is defined as $[A,B]_q = q^{1/2}AB - q^{-1/2}BA$. The algebra $U_q'(\mathrm{so}_3)$ can be defined as the associative algebra generated by the elements I_{21}, I_{32}, I_{31} satisfying relations (11).

Note that the algebra $U_q'(\mathrm{so}_3)$ has a big automorphism group. In fact, it is seen from (9) and (10) that these relations are not changed if we permute I_{21} and I_{32}. From relations (11), we see that the set of these relations are not changed under cyclical permutation of I_{21}, I_{32}, I_{31}. The change of a sign at I_{21} or at I_{32} also does not change relations (9) and (10). Generating a group by these automorphisms, we may find that they generate the group isomorphic to the modular group $SL(2,\mathbb{Z})$. That is why the algebra $U_q'(\mathrm{so}_3)$ is interesting for algebraic topology and algebraic geometry (see [26]–[28]).

Example 2. Let us consider the case of the algebra $U_q'(\mathrm{so}_4)$. It is generated by I_{21}, I_{32}, and I_{43}. We create the elements

$$I_{31}=[I_{21},I_{32}]_q,\ I_{42}=[I_{32},I_{43}]_q,\ I_{41}=[I_{21},I_{42}]_q. \tag{12}$$

Then the elements I_{ij}, $i>j$, satisfy the relations

$$[I_{21},I_{32}]_q=I_{31},\quad [I_{32},I_{31}]_q=I_{21},\quad [I_{31},I_{21}]_q=I_{32}.$$

$$[I_{32},I_{43}]_q=I_{42},\quad [I_{43},I_{42}]_q=I_{32},\quad [I_{42},I_{32}]_q=I_{43}.$$

$$[I_{31},I_{43}]_q=I_{41},\quad [I_{43},I_{41}]_q=I_{31},\quad [I_{41},I_{31}]_q=I_{43}.$$

$$[I_{21},I_{42}]_q=I_{41},\quad [I_{42},I_{41}]_q=I_{21},\quad [I_{41},I_{21}]_q=I_{42}.$$

$$[I_{21},I_{43}]=0,\quad [I_{32},I_{41}]=0,$$

$$[I_{42},I_{31}]=(q-q^{-1})(I_{21}I_{43}-I_{32}I_{41}).$$

At $q=1$, these relations define just the Lie algebra so(4). Each of the sets (I_{21},I_{32},I_{31}), (I_{32},I_{43},I_{42}), (I_{31},I_{43},I_{41}), (I_{21},I_{42},I_{41}) determines a subalgebra isomorphic to $U_q'(\mathrm{so}_3)$.

3. Irreducible Finite Dimensional Representations

The algebra $U_q'(\mathrm{so}_n)$ has two types of irreducible finite dimensional representations:

(a) representations of the classical type;

(b) representations of the nonclassical type.

Irreducible representations of the classical type are q-deformations of the irreducible finite dimensional representations of the Lie algebra so(n). So, there is a one-to-one correspondence between irreducible representations of the classical type of the algebra $U_q'(\mathrm{so}_n)$ and irreducible finite dimensional representations of so(n). Moreover, the formulas for representations of the classical type of $U_q'(\mathrm{so}_n)$ turn into the corresponding formulas for the representations of Lie algebra so(n) at $q\to 1$.

There exists no classical analogue for representations of the nonclassical type. Operators $T(a)$, $a\in U_q'(\mathrm{so}_n)$, have singularities at $q=1$. Let us describe irreducible finite dimensional representations of both types.

4. Irreducible Representations of the Classical Types

In this section, we describe (in the framework of the q-analogue of the Gel'fand–Tsetlin formalism) irreducible finite dimensional representations of the algebras $U_q'(\mathrm{so}_n)$, $n \geq 3$, which are q-deformations of the finite dimensional irreducible representations of the Lie algebra $\mathrm{so}(n)$. They are given by sets $\mathbf{m}_n$ of $\lfloor n/2 \rfloor$ numbers $m_{1,n}, m_{2,n}, ..., m_{\lfloor n/2 \rfloor,n}$ (here $\lfloor n/2 \rfloor$ denotes the integral part of $n/2$) which are all integral or all half-integral and satisfy the dominance conditions

$$m_{1,2p+1} \geq m_{2,2p+1} \geq ... \geq m_{p,2p+1} \geq 0,$$

$$m_{1,2p} \geq m_{2,2p} \geq ... \geq m_{p-1,2p} \geq |m_{p,2p}|$$

for $n = 2p+1$ and $n = 2p$, respectively. These representations are denoted by $T_{\mathbf{m}_n}$. We take the q-analogue of the Gel'fand–Tsetlin basis in the representation space, which is obtained by successive reduction of the representation $T_{\mathbf{m}_n}$ to the subalgebras $U_q'(\mathrm{so}_{n-1})$, $U_q'(\mathrm{so}_{n-2})$, $\cdots$, $U_q'(\mathrm{so}_3)$, $U_q'(\mathrm{so}_2) := U(\mathrm{so}_2)$. As in the classical case, its elements are labelled by the Gel'fand–Tsetlin tableaux

$$\{\xi_n\} \equiv \left\{ \begin{matrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \dots \\ \mathbf{m}_2 \end{matrix} \right\}, \qquad (13)$$

where the components of $\mathbf{m}_k$ and $\mathbf{m}_{k-1}$ satisfy the "betweenness" conditions

$$m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq ... \geq$$

$$\geq m_{p,2p+1} \geq m_{p,2p} \geq -m_{p,2p+1},$$

$$m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq ... \geq$$

$$\geq m_{p-1,2p-1} \geq |m_{p,2p}|.$$

The basis element defined by the tableau $\{\xi_n\}$ is denoted as $|\{\xi_n\}\rangle$ or simply as $|\xi_n\rangle$.

It is convenient to introduce the so-called l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \quad l_{j,2p} = m_{j,2p} + p - j,$$

for the numbers $m_{i,k}$. In particular, $l_{1,3} = m_{1,3} + 1$ and $l_{1,2} = m_{1,2}$. The operator $T_{\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\mathbf{m}_n}$ of $U_q'(\mathrm{so}_n)$ acts upon Gel'fand–Tsetlin basis elements, labelled by (13), by the formula

$$T_{\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^{p} \frac{A_{2p}^j(\xi_n)}{q^{l_{j,2p}} + q^{-l_{j,2p}}} |(\xi_n)_{2p}^{+j}\rangle -$$

$$-\sum_{j=1}^{p} \frac{A_{2p}^j((\xi_n)_{2p}^{-j})}{q^{l_{j,2p}} + q^{-l_{j,2p}}} |(\xi_n)_{2p}^{-j}\rangle \qquad (14)$$

and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$T_{\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1} \frac{B_{2p-1}^j(\xi_n)}{[2l_{j,2p-1}-1][l_{j,2p-1}]} |(\xi_n)_{2p-1}^{+j}\rangle$$

$$-\sum_{j=1}^{p-1} \frac{B_{2p-1}^j((\xi_n)_{2p-1}^{-j})}{[2l_{j,2p-1}-1][l_{j,2p-1}-1]} |(\xi_n)_{2p-1}^{-j}\rangle +$$

$$+\mathrm{i}\, C_{2p-1}(\xi_n)|\xi_n\rangle. \qquad (15)$$

In these formulas, $(\xi_n)_k^{\pm j}$ means tableau (13) in which the j-th component $m_{j,k}$ in $\mathbf{m}_k$ is replaced by $m_{j,k} \pm 1$. The coefficients A_{2p}^j, B_{2p-1}^j, C_{2p-1} in (14) and (15) are given by the expressions

$$A_{2p}^j(\xi_n) = \left(\frac{\prod_{i=1}^{p}[l_{i,2p+1}+l_{j,2p}][l_{i,2p+1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}][l_{i,2p}-l_{j,2p}]} \times \right.$$

$$\left. \times \frac{\prod_{i=1}^{p-1}[l_{i,2p-1}+l_{j,2p}][l_{i,2p-1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}+1][l_{i,2p}-l_{j,2p}-1]} \right)^{1/2}, \qquad (16)$$

$$B_{2p-1}^j(\xi_n) = \left(\frac{\prod_{i=1}^{p}[l_{i,2p}+l_{j,2p-1}][l_{i,2p}-l_{j,2p-1}]}{\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}][l_{i,2p-1}-l_{j,2p-1}]} \times \right.$$

$$\left. \times \frac{\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}][l_{i,2p-1}-l_{j,2p-1}]}{\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}-1][l_{i,2p-1}-l_{j,2p-1}-1]} \right)^{1/2}, \qquad (17)$$

$$C_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p}[l_{s,2p}]\prod_{s=1}^{p-1}[l_{s,2p-2}]}{\prod_{s=1}^{p-1}[l_{s,2p-1}][l_{s,2p-1}-1]}, \qquad (18)$$

where numbers in square brackets mean q-numbers defined by

$$[a] := \frac{q^a - q^{-a}}{q - q^{-1}}.$$

In particular,

$$T_{\mathbf{m}_n}(I_{3,2})|\xi_n\rangle =$$

$$= \frac{1}{q^{m_{1,2}}+q^{-m_{1,2}}} \left(\sqrt{[l_{1,3}+m_{1,2}][l_{1,3}-m_{1,2}-1]}\, |(\xi_n)_2^{+1}\rangle - \right.$$

$$\left. - \sqrt{[l_{1,3}+m_{1,2}-1][l_{1,3}-m_{1,2}]}\, |(\xi_n)_2^{-1}\rangle \right),$$

$T_{\mathbf{m}_n}(I_{2,1})|\xi_n\rangle = \mathrm{i}[m_{1,2}]|\xi_n\rangle,$

It is seen from formula (28) that the coefficient C_{2p-1} vanishes if $m_{p,2p} \equiv l_{p,2p} = 0$.

A proof of the fact that formulas (14)-(18) indeed determine a representation of $U'_q(\mathrm{so}_n)$ is given in [11].

Theorem 1. *The representations $T_{\mathbf{m}_n}$ are irreducible. The representations $T_{\mathbf{m}_n}$ and $T_{\mathbf{m}'_n}$ are pairwise nonequivalent for $\mathbf{m}_n \neq \mathbf{m}'_n$.*

Example 3. Irreducible representations of the classical type of the algebra $U'_q(\mathrm{so}_3)$ are given by a nonnegative integral or half-integral number l and act on vector spaces $\mathcal{H}_l$ with a basis $|l,m\rangle$, $m=-l,-l+1,\cdots,l$. We denote these representations by T_l. For the operators $T_l(I_{21})$ and $T_l(I_{32})$, we have the formulas $T_l(I_{21})l,m\rangle = \mathrm{i}[m]|l,m\rangle$ and

$$T_l(I_{32})l,m\rangle = \frac{1}{q^m+q^{-m}}(\sqrt{[l-m][l+m+1]}|l,m+1\rangle -$$

$$-\sqrt{[l+m][l-m+1]}l,m-1\rangle),$$

where $[a]$ denotes a q-number.

5. Irreducible Representations of the Nonclassical Types

Irreducible finite dimensional representations of the nonclassical type are given by sets $\epsilon := (\epsilon_2,\epsilon_3,\cdots,\epsilon_n)$, $\epsilon_i = \pm 1$, and by sets $\mathbf{m}_n$ consisting of $\lfloor n/2 \rfloor$ **half-integral** numbers $m_{1,n}, m_{2,n}, \cdots, m_{\lfloor n/2\rfloor,n}$ that satisfy the dominance conditions

$$m_{1,2p+1} \geq m_{2,2p+1} \geq \ldots \geq m_{p,2p+1} \geq 1/2,$$

$$m_{1,2p} \geq m_{2,2p} \geq \ldots \geq m_{p-1,2p} \geq m_{p,2p} \geq 1/2$$

for $n=2p+1$ and $n=2p$, respectively. These representations are denoted by $T_{\epsilon,\mathbf{m}_n}$.

For a basis in the representation space, we use the analogue of the basis of the previous section. Its elements are labelled by tableaux

$$\{\xi_n\} \equiv \left\{ \begin{matrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \ldots \\ \mathbf{m}_2 \end{matrix} \right\}, \tag{19}$$

where the components of $\mathbf{m}_k$ and $\mathbf{m}_{k-1}$ satisfy the "betweenness" conditions

$$m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq \ldots \geq$$

$$\geq m_{p,2p+1} \geq m_{p,2p} \geq 1/2,$$

$$m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq \ldots \geq$$

$$\geq m_{p-1,2p-1} \geq m_{p,2p}.$$

The basis element defined by the tableau $\{\xi_n\}$ is denoted as $|\xi_n\rangle$.

As in the previous section, it is convenient to introduce the l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \quad l_{j,2p} = m_{j,2p} + p - j.$$

The operator $T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\epsilon,\mathbf{m}_n}$ of $U_q(\mathrm{so}_n)$ acts upon our basis elements, labelled by (19), by the formulas

$$T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \delta_{m_{p,2p},1/2}\frac{\epsilon_{2p+1}}{q^{1/2}-q^{-1/2}}D_{2p}(\xi_n)|\xi_n\rangle$$

$$+\sum_{j=1}^{p}\frac{A^j_{2p}(\xi_n)}{q^{l_{j,2p}}-q^{-l_{j,2p}}}|(\xi_n)^{+j}_{2p}\rangle - \sum_{j=1}^{p}\frac{A^j_{2p}((\xi_n)^{-j}_{2p})}{q^{l_{j,2p}}-q^{-l_{j,2p}}}|(\xi_n)^{-j}_{2p}\rangle, \tag{20}$$

where the summation in the last sum must be performed from 1 to $p-1$ if $m_{p,2p} = 1/2$, and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$T_{\epsilon,\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1}\frac{B^j_{2p-1}(\xi_n)}{[2l_{j,2p-1}-1][l_{j,2p-1}]_+}|(\xi_n)^{+j}_{2p-1}\rangle$$

$$-\sum_{j=1}^{p-1}\frac{B^j_{2p-1}((\xi_n)^{-j}_{2p-1})}{[2l_{j,2p-1}-1][l_{j,2p-1}-1]_+}|(\xi_n)^{-j}_{2p-1}\rangle +$$

$$+\epsilon_{2p}\hat{C}_{2p-1}(\xi_n)|\xi_n\rangle, \tag{21}$$

where $[a]_+ = (q^a+q^{-a})/(q-q^{-1})$. In these formulas, $(\xi_n)^{\pm j}_k$ means tableau (19) in which the j-th component $m_{j,k}$ in $\mathbf{m}_k$ is replaced by $m_{j,k}\pm 1$. The matrix elements A^j_{2p} and B^j_{2p-1} are given by the same formulas as in (14) and (15) (that is, by formulas (16) and (17)) and

$$\hat{C}_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p}[l_{s,2p}]_+ \prod_{s=1}^{p-1}[l_{s,2p-2}]_+}{\prod_{s=1}^{p-1}[l_{s,2p-1}]_+[l_{s,2p-1}-1]_+},$$

$$D_{2p}(\xi_n) = \frac{\prod_{i=1}^{p}[l_{i,2p+1}-\frac{1}{2}]\prod_{i=1}^{p-1}[l_{i,2p-1}-\frac{1}{2}]}{\prod_{i=1}^{p-1}[l_{i,2p}+\frac{1}{2}][l_{i,2p}-\frac{1}{2}]}.$$

For the operators $T_{\epsilon,\mathbf{m}_n}(I_{3,2})$ and $T_{\epsilon,\mathbf{m}_n}(I_{2,1})$, we have

$$T_{\epsilon,\mathbf{m}_n}(I_{3,2})|\xi_n\rangle =$$

$$=\frac{1}{q^{m_{1,2}}-q^{-m_{1,2}}}\left(\sqrt{[l_{1,3}+m_{1,2}][l_{1,3}-m_{1,2}-1]}|(\xi_n)_2^{+1}\rangle-\right.$$

$$\left.-\sqrt{[l_{1,3}+m_{1,2}-1][l_{1,3}-m_{1,2}]}|(\xi_n)_2^{-1}\rangle\right)$$

if $m_{1,2}\ne \frac{1}{2}$,

$$T_{\epsilon,\mathbf{m}_n}(I_{3,2})|\xi_n\rangle=\frac{1}{q^{1/2}-q^{-1/2}}\left(\epsilon_3[l_{1,3}-1/2]|(\xi_n)\rangle+\right.$$

$$\left.+\sqrt{[l_{1,3}+1/2][l_{1,3}-3/2]}|(\xi_n)_2^{+1}\rangle\right)$$

if $m_{1,2}=\frac{1}{2}$, and $T_{\epsilon,\mathbf{m}_n}(I_{2,1})|\xi_n\rangle=\epsilon_2[m_{1,2}]_+|\xi_n\rangle$.

The fact that the above operators $T_{\epsilon,\mathbf{m}_n}(I_{k,k-1})$ satisfy the defining relations (1)–(3) of the algebra $U'_q(\mathrm{so}_n)$ is proved [12] in the following way. We take formulas (15)–(18) for the classical type representations $T_{\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ with half-integral $m_{i,n}$ and replace there every $m_{j,2p+1}$ by $m_{j,2p+1}-\mathrm{i}\pi/2h$, every $m_{j,2p}$, $j\ne p$, by $m_{j,2p}-\mathrm{i}\pi/2h$ and $m_{p,2p}$ by $m_{p,2p}-\epsilon_2\epsilon_4\cdots\epsilon_{2p}\mathrm{i}\pi/2h$, where each ϵ_{2s} is equal to $+1$ or -1 and h is defined by $q=e^h$. Repeating almost word by word the reasoning of paper [11], we prove that the operators given by formulas (15)–(18) satisfy the defining relations (1)–(3) of the algebra $U'_q(\mathrm{so}_n)$ after this replacement. Therefore, these operators determine a representation of $U'_q(\mathrm{so}_n)$. We denote this representation by $T'_{\mathbf{m}_n}$. After a simple rescaling, the operators $T'_{\mathbf{m}_n}(I_{k,k-1})$ take the form

$$T'_{\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle=\sum_{j=1}^{p}\frac{A^j_{2p}(\xi_n)}{q^{l_{j,2p}}-q^{-l_{j,2p}}}|(\xi_n)^{+j}_{2p}\rangle-$$

$$-\sum_{j=1}^{p}\frac{A^j_{2p}((\xi_n)^{-j}_{2p})}{q^{l_{j,2p}}-q^{-l_{j,2p}}}|(\xi_n)^{-j}_{2p}\rangle,$$

$$T'_{\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle=\sum_{j=1}^{p-1}\frac{B^j_{2p-1}(\xi_n)}{[2l_{j,2p-1}-1][l_{j,2p-1}]_+}|(\xi_n)^{+j}_{2p-1}\rangle-$$

$$-\sum_{j=1}^{p-1}\frac{B^j_{2p-1}((\xi_n)^{-j}_{2p-1})}{[2l_{j,2p-1}-1][l_{j,2p-1}-1]_+}|(\xi_n)^{-j}_{2p-1}\rangle+$$

$$+\epsilon_{2p}\hat{C}_{2p-1}(\xi_n)|\xi_n\rangle,$$

where A^j_{2p}, B^j_{2p-1} and $\hat{C}_{2p-1}$ are such as in formulas (20) and (21). The representations $T'_{\mathbf{m}_n}$ are reducible. We decompose these representations into subrepresentations in the following way. We fix p ($p=1,2,\cdots,\lfloor(n-1)/2\rfloor$) and decompose the space $\mathcal{H}$ of the representation $T'_{\mathbf{m}_n}$ into a direct sum of two subspaces $\mathcal{H}_{\epsilon_{2p+1}}$, $\epsilon_{2p+1}=\pm1$, spanned by the basis vectors

$$|\xi_n\rangle_{\epsilon_{2p+1}}=|\xi_n\rangle-\epsilon_{2p+1}|\xi'_n\rangle,\qquad m_{p,2p}\ge 1/2,$$

respectively, where $|\xi'_n\rangle$ is obtained from $|\xi_n\rangle$ by the replacement of $m_{p,2p}$ by $-m_{p,2p}$. A direct verification shows that two subspaces $\mathcal{H}_{\epsilon_{2p+1}}$ are invariant with respect to all the operators $T'_{\mathbf{m}_n}(I_{k,k-1})$. Now we take the subspaces $\mathcal{H}_{\epsilon_{2p+1}}$ and repeat the same procedure for some s, $s\ne p$, and decompose each of these subspaces into two invariant subspaces. Continuing this procedure further, we decompose the representation space $\mathcal{H}$ into a direct sum of $2^{\lfloor(n-1)/2\rfloor}$ invariant subspaces. The operators $T'_{\mathbf{m}_n}(I_{k,k-1})$ act upon these subspaces according to formulas (20) and (21). We denote the corresponding subrepresentations on these subspaces by $T_{\epsilon,\mathbf{m}_n}$. The above reasoning shows that the operators $T_{\epsilon,\mathbf{m}_n}(I_{k,k-1})$ satisfy the defining relations (1)–(3) of the algebra $U'_q(\mathrm{so}_n)$.

Theorem 2. *The representations $T_{\epsilon,\mathbf{m}_n}$ are irreducible. The representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\epsilon',\mathbf{m}'_n}$ are pairwise nonequivalent for $(\epsilon,\mathbf{m}_n)\ne(\epsilon',\mathbf{m}'_n)$. For any admissable $(\epsilon,\mathbf{m}_n)$ and $\mathbf{m}'_n$, the representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\mathbf{m}'_n}$ are pairwise nonequivalent.*

The algebra $U'_q(\mathrm{so}_n)$ has non-trivial one-dimensional representations. They are special cases of the representations of the nonclassical type. They are described as follows.

Let $\epsilon:=(\epsilon_2,\epsilon_3,\cdots,\epsilon_n)$, $\epsilon_i=\pm1$, and let $\mathbf{m}_n=(m_{1,n},m_{2,n},\cdots,m_{\lfloor n/2\rfloor,n})=(\frac12,\frac12,\cdots,\frac12)$. Then the corresponding representations $T_{\epsilon,\mathbf{m}_n}$ are one-dimensional and are given by the formulas

$$T_{\epsilon,\mathbf{m}_n}(I_{k+1,k})|\xi_n\rangle=\epsilon_{k+1}(q^{1/2}-q^{-1/2})^{-1}.$$

Thus, to every $\epsilon:=(\epsilon_2,\epsilon_3,\cdots,\epsilon_n)$, $\epsilon_i=\pm1$, there corresponds a one-dimensional representation of $U'_q(\mathrm{so}_n)$.

Example 4. Let us describe irreducible representations of the nonclassical type of the algebra $U'_q(\mathrm{so}_3)$ These representations are given by numbers m,ϵ_1,ϵ_2, where m is a positive half-integer and $\epsilon_1,\epsilon_2=\pm1$. We replace m by the number $k=m+1/2$. Then k runs over positive integers. The corresponding representation of $U'_q(\mathrm{so}_3)$ is denoted by $R_k^{\epsilon_1,\epsilon_2}$. The basis vectors of the representation space are $|r\rangle$, $r=1,2,\cdots,k$. For the operators of the representation $R_k^{\epsilon_1,\epsilon_2}$, we have

$$R_k^{\epsilon_1,\epsilon_2}(I_{21})|r\rangle=\epsilon_1\frac{q^{r-1/2}+q^{-r+1/2}}{q-q^{-1}}|r\rangle,$$

$$R_k^{\epsilon_1,\epsilon_2}(I_{32})|1\rangle = \frac{1}{q^{1/2}-q^{-1/2}}(\epsilon_2[k]_q|1\rangle + \mathrm{i}[k-1]_q|2\rangle),$$

$$R_k^{\epsilon_1,\epsilon_2}(I_{32})|r\rangle = \frac{1}{q^{r-1/2}-q^{-r+1/2}}(\mathrm{i}[k-r]_q|r+1\rangle +$$

$$+\mathrm{i}[k+r-1]_q|r-1\rangle).$$

It is easy to see from these formulas that Tr $R_k^{\epsilon_1,\epsilon_2}(I_{21}) \neq 0$ and Tr $R_k^{\epsilon_1,\epsilon_2}(I_{32}) \neq 0$. Note that, for representations of the classical type, the elements I_{21} and I_{32} correspond to operators with vanishing trace.

6. Finite Dimensionality of Representations for q a Root of Unity

Everywhere below in this and the following sections, we assume that q is a root of unity. Moreover, we consider that $q^k = 1$ and k is an odd integer.

We shall need an information on the center of the algebra $U'_q(\mathrm{so}_n)$. Central elements of the algebra $U'_q(\mathrm{so}_n)$ for any value of q are described in [24] and [29]. They are given in the form of homogeneous polynomials of elements of $U'_q(\mathrm{so}_n)$. If q is a root of unity, then there exist additional central elements of $U'_q(\mathrm{so}_n)$ which are given by the following theorem proved in [16].

Theorem 3. *Let $q^k = 1$ for $k \in \mathbb{N}$ and $q^j \neq 1$ for $0 < j < k$. Then all the elements*

$$C^{(k)}(I_{rl}^+) = \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{k-j}{j} \frac{1}{k-j} \Big(\frac{i}{q-q^{-1}}\Big)^{2j} I_{rl}^{+\,k-2j}$$

for $r > l$, where $\lfloor (k-1)/2 \rfloor$ is the integral part of the number $(k-1)/2$, belong to the center of $U'_q(\mathrm{so}_n)$.

It is well known that a Drinfeld–Jimbo algebra $U_q(g)$ for q a root of unity ($q^k = 1$) is a finite dimensional vector space over the center of $U_q(g)$. The same assertion is true for the algebra $U'_q(\mathrm{so}_n)$. In fact, any element $(I_{ij}^+)^s$, $s \geq k$, can be reduced to a linear combination of $(I_{ij}^+)^r$, $r < k$, with coefficients from the center $\mathcal{C}$ of $U'_q(\mathrm{so}_n)$. Now our assertion follows from this sentence and from the Poincaré–Birkhoff–Witt theorem for $U'_q(\mathrm{so}_n)$.

Theorem 4. *If q is a root of unity, then any irreducible representation of $U'_q(\mathrm{so}_n)$ is finite dimensional.*

This theorem is proved in the following way. Let q be a root of unity, that is, $q^k = 1$. Let T be an irreducible representation of $U'_q(\mathrm{so}_n)$ on a vector space $\mathcal{V}$. Then T maps central elements into scalar operators. Since the linear space $U'_q(\mathrm{so}_n)$ is finite dimensional over the center $\mathcal{C}$ with the basis $I_{21}^{+\,m_{21}} I_{31}^{+\,m_{31}} \cdots I_{n,n-1}^{+\ \ m_{n,n-1}}$, $m_{ij} < k$, we have $T(a) = \sum_{m_{ij}<k} c_{\{m_{ij}\}} T(I_{21}^{+\,m_{21}} I_{31}^{+\,m_{31}} \cdots I_{n,n-1}^{+\ \ m_{n,n-1}})$ for any $a \in U'_q(\mathrm{so}_n)$, where $c_{\{m_{ij}\}}$ are numerical coefficients. Hence, if $\mathbf{v}$ is a nonzero vector of the representation space $\mathcal{V}$, then $T(U'_q(\mathrm{so}_n))\mathbf{v} = \mathcal{V}$ since T is an irreducible representation. Since $T(a)$ is of the above form for any $a \in U'_q(\mathrm{so}_n)$, $\mathcal{V}$ is finite dimensional. Theorem 4 is proved.

It follows from this proof that there exists a fixed positive integer r such that the dimension of any irreducible representation of $U'_q(\mathrm{so}_n)$ at q a root of unity does not exceed r. Of course, the number r depends on k.

7. Irreducible Representations at q a Root of Unity

Let us consider irreducible representations of $U'_q(\mathrm{so}_n)$ for q a root of unity ($q^k = 1$ and k is the smallest positive integer with this property). We also assume that k is odd. If k would be even, then almost all below reasoning is true, if k is replaced by $k' = k/2$ (as in the case of irreducible representations of the quantum algebra $U_q(\mathrm{sl}_2)$ for q a root of unity in [4], chapter 3).

There are many series of irreducible representations of $U'_q(\mathrm{so}_n)$ in this case. We describe the main series of such representations. We fix complex numbers $m_{1,n}, m_{2,n}, ..., m_{\lfloor n/2 \rfloor,n}$ (here $\lfloor n/2 \rfloor$ denotes the integral part of $n/2$) and c_{ij}, h_{ij}, $j = 2, 3, \cdots, n-1$, $i = 1, 2, \cdots, \lfloor j/2 \rfloor$ such that none of the numbers

$$m_{in},\ \ h_{ij},\ \ h_{ij}-h_{sj},\ \ h_{ij}-h_{s,j\pm1},\ \ h_{ij}+h_{sj},$$

$$h_{ij}+h_{s,j\pm1},\ \ h_{b,n-1}-m_{sn},\ \ h_{b,n-1}+m_{sn}$$

belongs to $\frac{1}{2}\mathbb{Z}$. (We also suppose that $c_{ij} \neq 0$.) The set of these numbers will be denoted by ω:

$$\omega = \{\mathbf{m}_n, \mathbf{c}_{n-1}, \mathbf{h}_{n-1}, \cdots, \mathbf{c}_2, \mathbf{h}_2\},$$

where $\mathbf{m}_n$ is the set of numbers $m_{1,n}, m_{2,n}, ..., m_{\lfloor n/2 \rfloor,n}$, and, $\mathbf{c}_j$ and $\mathbf{h}_j$ are the sets of numbers c_{ij}, $i = 1, 2, \cdots, \lfloor j/2 \rfloor$, and h_{ij}, $i = 1, 2, \cdots, \lfloor j/2 \rfloor$, respectively. (Thus, ω contains $r = \dim \mathrm{so}_n$ complex numbers.) Let V be a complex vector space with a basis labelled by the tableaux

$$\{\xi_n\} \equiv \left\{ \begin{matrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \cdots \\ \mathbf{m}_2 \end{matrix} \right\}, \qquad (22)$$

where the set of numbers $\mathbf{m}_n$ consists of $\lfloor n/2 \rfloor$ numbers $m_{1,n}, m_{2,n}, \cdots, m_{\lfloor n/2 \rfloor,n}$ given above, and, for each $s = 2, 3, \cdots, n-1$, $\mathbf{m}_s$ is a set of numbers $m_{1,s}, \cdots, m_{\lfloor s/2 \rfloor,s}$ and each $m_{i,s}$ runs independently the values $h_{i,s}, h_{i,s}+1, \cdots, h_{i,s}+k-1$. *Thus,* $\dim V$ *coincides with* k^N, *where* N *is the number of positive roots of* $\mathrm{so}(n)$. It is convenient to use the so-called l-coordinates for the numbers $m_{i,s}$, $s = 2, 3, \cdots, n$:

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \quad l_{j,2p} = m_{j,2p} + p - j.$$

To the set of numbers ω, there corresponds the irreducible finite dimensional representation T_ω of the algebra $U'_q(\mathrm{so}_n)$. The operators $T_\omega(I_{2p+1,2p})$ of the representation T_ω act upon the basis elements, labelled by (32), by the formula

$$T_\omega(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^{p} c_{j,2p} \frac{A^j_{2p}(\xi_n)}{q^{l_{j,2p}} + q^{-l_{j,2p}}} |(\xi_n)^{+j}_{2p}\rangle -$$

$$- \sum_{j=1}^{p} c^{-1}_{j,2p} \frac{A^j_{2p}((\xi_n)^{-j}_{2p})}{q^{l_{j,2p}} + q^{-l_{j,2p}}} |(\xi_n)^{-j}_{2p}\rangle \qquad (23)$$

and the operators $T_\omega(I_{2p,2p-1})$ of the representation T_ω act as

$$T_\omega(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1} \frac{c_{j,2p-1} B^j_{2p-1}(\xi_n)}{[2l_{j,2p-1}-1][l_{j,2p-1}]} |(\xi_n)^{+j}_{2p-1}\rangle -$$

$$- \sum_{j=1}^{p-1} c^{-1}_{j,j,2p-1} \frac{c^{-1}_{j,j,2p-1} B^j_{2p-1}((\xi_n)^{-j}_{2p-1})}{[2l_{j,2p-1}-1][l_{j,2p-1}-1]} |(\xi_n)^{-j}_{2p-1}\rangle +$$

$$+ \mathrm{i}\, C_{2p-1}(\xi_n)\, |\xi_n\rangle, \qquad (24)$$

where numbers in square brackets mean q-numbers. In these formulas, $(\xi_n)^{\pm j}_s$ means tableau (22), in which j-th component $m_{j,s}$ in $\mathbf{m}_s$ is replaced by $m_{j,s} \pm 1$. If $m_{j,s}+1 = h_{j,s}+k$ (resp. $m_{j,s}-1 = h_{j,s}-1$), then we set $m_{j,s}+1 = h_{j,s}$ (resp. $m_{j,s}-1 = h_{j,s}+k-1$). The coefficients A^j_{2p}, B^j_{2p-1}, C_{2p-1} in (23) and (24) are given by the expressions

$$A^j_{2p}(\xi_n) = \left(\frac{\prod_{i=1}^{p}[l_{i,2p+1}+l_{j,2p}][l_{i,2p+1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}][l_{i,2p}-l_{j,2p}]} \times \right.$$

$$\left. \times \frac{\prod_{i=1}^{p-1}[l_{i,2p-1}+l_{j,2p}][l_{i,2p-1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}+1][l_{i,2p}-l_{j,2p}-1]} \right)^{1/2},$$

$$B^j_{2p-1}(\xi_n) = \left(\frac{\prod_{i=1}^{p}[l_{i,2p}+l_{j,2p-1}][l_{i,2p}-l_{j,2p-1}]}{\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}][l_{i,2p-1}-l_{j,2p-1}]} \times \right.$$

$$\left. \times \frac{\prod_{i=1}^{p-1}[l_{i,2p-2}+l_{j,2p-1}][l_{i,2p-2}-l_{j,2p-1}]}{\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}-1][l_{i,2p-1}-l_{j,2p-1}-1]} \right)^{1/2},$$

$$C_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p}[l_{s,2p}] \prod_{s=1}^{p-1}[l_{s,2p-2}]}{\prod_{s=1}^{p-1}[l_{s,2p-1}][l_{s,2p-1}-1]}.$$

The fact that the operators $T_\omega(I_{j,j-1})$, given above, satisfy the defining relations (1)–(3) is proved in the same way as in the case of irreducible representations of $U'_q(\mathrm{so}_n)$ when q is not a root of unity (see [15]).

As in the case of finite dimensional irreducible representations of the Lie algebra so_n, the form of the basis elements of the above representation space V and the formulas for the operators $T_\omega(I_{j,j-1})$ allow us to decompose the restriction of the representation T_ω, $\omega = \{\mathbf{m}_n, \mathbf{c}_{n-1}, \mathbf{h}_{n-1}, \cdots, \mathbf{c}_2, \mathbf{h}_2\}$, to the subalgebra $U'_q(\mathrm{so}_{n-1})$. We have

$$T_\omega\big|_{U'_q(\mathrm{so}_{n-1})} = \bigoplus\nolimits_{\omega_{n-1}} T_{\omega_{n-1}},$$

where $\omega_{n-1} = \{\mathbf{m}_{n-1}, \mathbf{c}_{n-2}, \mathbf{h}_{n-2}, \cdots, \mathbf{c}_2, \mathbf{h}_2\}$ and $\mathbf{m}_{n-1}$ runs over the vectors

$$(h_{1,n-1}+a_1, h_{2,n-1}+a_2, \cdots, h_{s,n-1}+a_s),$$

$$s = \lfloor (n-1)/2 \rfloor, \quad a_j = 0, 1, 2, \cdots, k-1,$$

and $\mathbf{c}_j$ and $\mathbf{h}_j$ are such as in ω.

Theorem 5 [15]. *Representations* T_ω *with the domain of values of representation parameters, as described above, are irreducible.*

There are equivalence relations in the set of irreducible representations T_ω. In order to extract a subset of pairwise nonequivalent representations from the entire set, we introduce some domains on the complex plane. The set

$$D = \{x \in \mathbb{C} \mid |\mathrm{Re}\, x| < k/4 \text{ or } \mathrm{Re}\, x = -k/4,$$

$$\mathrm{Im}\, x \le 0 \text{ or } \mathrm{Re}\, x = k/4,\ \mathrm{Im}\, x \ge 0\}$$

is a maximal subset of $\mathbb{C}$ such that, for all $x, y \in D$, $x \neq y$, we have $[x] \neq [y]$. The set

$$D^{\pm} = \{x \in \mathbb{C} \mid 0 < \mathrm{Re}\, x < k/4 \text{ or } \mathrm{Re}\, x = 0,$$

$$\mathrm{Im}\, x \ge 0 \text{ or } \mathrm{Re}\, x = k/4,\ \mathrm{Im}\, x \ge 0\}$$

is a maximal subset of $\mathbb{C}$ such that, for all $x, y \in D^{\pm}$, $x \neq y$, we have $[x] \neq \pm[y]$. We need also the sets

$$D_h = \{x \in \mathbb{C} \mid |\mathrm{Re}\, x| < 1/4 \text{ or } \mathrm{Re}\, x = -1/4,$$

$$\mathrm{Im}\, x \leq 0 \text{ or } \mathrm{Re}\, x = 1/4,\ \mathrm{Im}\, x \geq 0\},$$

$$D_h^{\pm} = \{x \in \mathbb{C} \mid 0 < \mathrm{Re}\, x < 1/4 \text{ or } \mathrm{Re}\, x = 0,$$

$$\mathrm{Im}\, x \geq 0 \text{ or } \mathrm{Re}\, x = 1/4,\ \mathrm{Im}\, x \geq 0\}.$$

We introduce an ordering in the set $D^{\pm}$ (resp. $D_h^{\pm}$) as follows: we say that $x \succ y$, $x, y \in D^{\pm}$ (resp., $x, y \in D_h^{\pm}$) if either $\mathrm{Re}\, x > \mathrm{Re}\, y$ or both $\mathrm{Re}\, x = \mathrm{Re}\, y$ and $\mathrm{Im}\, x > \mathrm{Im}\, y$.

We say that the set of complex numbers $\mathbf{l}_{2p} = (l_{1,2p}, l_{2,2p}, \cdots, l_{p,2p})$ is *dominant* if $l_{1,2p}$, $l_{2,2p}, \cdots$, $l_{p-1,2p} \in D^{\pm}$, $l_{p,2p} \in D$, and $l_{1,2p} \succ l_{2,2p} \succ \cdots \succ l_{p-1,2p} \succ l^*_{p,2p}$, where $l^*_{p,2p} = l_{p,2p}$ if $l_{p,2p} \in D^{\pm}$ and $l^*_{p,2p} = -l_{p,2p}$ if $l_{p,2p} \notin D^{\pm}$.

The notion of *dominance* for the set $\mathbf{h}_{2p} = (h_{1,2p}, h_{2,2p}, \cdots, h_{p,2p}) \in \mathbb{C}^p$ is introduced by the replacements $l_{i,2p} \to h_{i,2p}$, $D \to D_h$ and $D^{\pm} \to D_h^{\pm}$ in the previous definition.

We say that the set of complex numbers $\mathbf{l}_{2p+1} = (l_{1,2p+1}, l_{2,2p+1}, \cdots, l_{p,2p+1})$ is *dominant* if $l_{1,2p+1}$, $l_{2,2p+1}, \cdots, l_{p,2p+1} \in D^{\pm}$ and $l_{1,2p+1} \succ l_{2,2p+1} \succ \cdots \succ l_{p,2p+1}$.

The notion of *dominance* for the set of complex numbers $\mathbf{h}_{2p+1} = (h_{1,2p+1}, h_{2,2p+1}, \cdots, h_{p,2p+1})$ is introduced by the replacements $l_{i,2p+1} \to h_{i,2p+1}$ and $D^{\pm} \to D_h^{\pm}$ in the previous definition.

We say that $\omega = \{\mathbf{m}_n, \mathbf{c}_{n-1}, \mathbf{h}_{n-1}, \cdots, \mathbf{c}_2, \mathbf{h}_2\}$ is *dominant* if every of the sets $\mathbf{l}_n, \mathbf{h}_{n-1}, \cdots, \mathbf{h}_2$ is dominant and if $0 \leq \mathrm{Arg}\, c_{ij} < 2\pi/k$, $j = 2, 3, \cdots, n-1$; $i = 1, 2, \cdots, \lfloor j/2 \rfloor$.

Theorem 6 [15]. *The representations T_ω of $U'_q(\mathrm{so}_n)$ with dominant ω are pairwise nonequivalent. Any irreducible representation $T_{\omega'}$ is equivalent to some representation T_ω with dominant ω.*

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1. *Jimbo M.* // Lett. Math. Phys. – 1985. – **10**. – P. 63–69.
2. *Drinfeld V. G.* // Sov. Math. Dokl. – 1985. – **32**. – P. 354–258.
3. *Jantzen J. C.* Lectures on Quantum Groups. – Providence, RI: Amer. Math. Soc., 1996.
4. *Klimyk A., Schmüdgen K.* Quantum Groups and Their Representations. – Berlin: Springer, 1997.
5. *Gavrilik A. M., Klimyk A. U.*//Lett. Math. Phys. – 1991. – **21**. – P. 215–220.
6. *Fairlie D. B.*//J. Phys. A: Math. Gen. – 1990. – **23**. – P. L183–L187.
7. *Odesskii A.*//Funct. Anal. Appl. – 1986. – **20**. – P. 152–154.
8. *Havlíček M., Klimyk A. U., Pošta S.*//J. Math. Phys. – 2001. – **42**, N 11. – P. 5389–5416.
9. *Noumi M.*//Adv. Math. – 1996. – **123**, No. 1. – P. 16–77.
10. *Iorgov N. Z.*//J. Phys. A: Math. Gen. – 2001. – **34**. – P. 3095–3108.
11. *Gavrilik A. M., Iorgov N. Z.*//Methods of Funct. Anal. Topology – 1997. – **3**, No. 4. – P. 51–63.
12. *Iorgov N. Z., Klimyk A. U.*//Czech. J. Phys. – 2000. – **50**, N 1. – P. 85–90.
13. *Havlíček M., Klimyk A. U., Pošta S.*//J. Math. Phys. – 1999. – **40**. – P. 2135–2161.
14. *Samoilenko Yu. S., Turowska L.* //Quantum Groups and Quantum Spaces. – Warsaw: Banach Center Publications, 1997. – P. 21–43.
15. *Iorgov N. Z., Klimyk A. U.*//Meth. Funct. Anal. Topology – 1999. – **3**, N 6. – P. 56–73.
16. *Havlíček M., Klimyk A. U., Pošta S.*//Czech. J. Phys. – 2000. – **50**, N 1. – P. 79–84.
17. *Arnaudon D., Chakrabarti A.*//Communs. Math. Phys. – 1991. – **139**. – P. 461–478.
18. *Nelson J., Regge T., Zertuche F.*//Nucl. Phys. – 1990. – **B339**, N 1. – P. 227.
19. *Nelson J., Regge T.*//Communs. Math. Phys. – 1993. – **155**. – P. 561.
20. *Gavrilik A. M.*//Proc. Inst. Math. of Nat. Acad. Sci. of Ukraine. – 2000. – **30**, N 2. – P. 304–309.
21. *Iorgov N. Z., Klimyk A. U.*//J. Math. Phys. – 2001. – **42**, N 11. – P. 1137–1148.
22. *Iorgov N. Z., Klimyk A. U.*//Phys. Atom. Nucl. – 2001. – **64**, N 12. – P. 1973–1977.
23. *Klimyk A. U.*//Proc. Intern. Conf. "Quantum Theory and Symmetry". – Singapore: World Scientific, 2000. – P. 459–464.
24. *Noumi M., Umeda T., Wakayama M.*//Compos. Math. – 1996. – **104**. N 2. – P. 227–277.
25. *Gavrilik A. M., Iorgov N. Z.*//Ukr. J. Phys. – 1998. – **43**. N 7. – P. 791–797.
26. *Chekhov L. O., Fock V. V.*//Czech. J. Phys. – 2000. – **50**, N 11. – P. 1201–1208.
27. *Bullock D., Przytycky J. H.*//Multiplicative structure of Kauffman bracket skein module quantization, **e**-print: math.QA/9902117.
28. *Przytycky J. H.*//Proceeding of Knots 96 (S. Suzuki, Ed.). – Singapore: World Scientific, 1997. – P. 279–292.
29. *Gavrilik A. M., Iorgov N. Z.*//Heavy Ion Phys. – 2000. – **11**, N 1. – P. 29–32.

ПРЕДСТАВЛЕННЯ АЛГЕБРИ $U'_q(\mathrm{so}_n)$, ЗВ'ЯЗАНОЇ З КВАНТОВОЮ ГРАВІТАЦІЄЮ

А. У. Клімик

Р е з ю м е

Дана робота є оглядом наших результатів зі скінченновимірних представлень нестандартної q-деформації $U'_q(\mathrm{so}_n)$ універсальної обвідної алгебри $U(\mathrm{so}_n)$ алгебри Лі so_n, яка не збігається з квантовою алгеброю Дрінфельда–Джімбо $U_q(\mathrm{so}_n)$. Ця алгебра зв'язана з алгебрами спостережуваних у квантовій гравітації і з алгебраїчною геометрією. Наведено незвідні скінченновимірні представлення алгебри $U'_q(\mathrm{so}_n)$, коли q не є коренем з одиниці і коли q – корінь з одиниці.

ПРЕДСТАВЛЕНИЯ АЛГЕБРЫ $U'_q(\mathrm{so}_n)$, СВЯЗАННОЙ С КВАНТОВОЙ ГРАВИТАЦИЕЙ

А. У. Климык

Р е з ю м е

Данная работа представляет собой обзор наших результатов по конечномерным представлениям нестандартной q-деформации $U'_q(\mathrm{so}_n)$ универсальной обертывающей алгебры $U(\mathrm{so}_n)$ алгебры Ли so_n, не совпадающей с квантовой алгеброй Дринфельда–Джимбо $U_q(\mathrm{so}_n)$. Эта алгебра связана с алгебрами наблюдаемых в квантовой гравитации и с алгебраической геометрией. Приведены неприводимые конечномерные представления алгебры $U'_q(\mathrm{so}_n)$ для случаев, когда q не является корнем из единицы и когда q – корень из единицы.