

WIGNER–ECKART THEOREM FOR AN ALGEBRA RELATED TO QUANTUM GRAVITY¹

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The tensor product of vector and arbitrary representations of the nonstandard q -deformation $U'_q(\mathfrak{so}_n)$ of the universal enveloping algebra $U(\mathfrak{so}_n)$ of the Lie algebra \mathfrak{so}_n is defined. This algebra is known to be related to $(2+1)$ -dimensional quantum gravity. The Clebsch–Gordan coefficients of tensor product of vector and arbitrary representations of the classical or nonclassical type of the q -algebra $U'_q(\mathfrak{so}_n)$ are found in an explicit form. The Wigner–Eckart theorem for vector operators is proved.

1. Introduction

For the last fifteen years, much attention of mathematicians and mathematical physicists is attracted to the subject of quantum algebras and quantum groups. Besides the standard deformation of Lie algebras proposed by Drinfeld [1] and Jimbo [2], other (nonstandard) deformations are also under consideration. This paper deals with the deformation $U'_q(\mathfrak{so}_n)$ of the universal enveloping algebra $U(\mathfrak{so}_n)$ proposed by Gavrilik and Klimyk [3]. Let us mention that the algebra $U'_q(\mathfrak{so}_3)$ appeared earlier in [4].

As a matter of interest, the algebras $U'_q(\mathfrak{so}_n)$ arose naturally as auxiliary algebras in deriving the algebra of observables in $2+1$ quantum gravity with $2D$ space of genus g , so that n depends on g , $n = 2g + 2$ [5 – 7].

As shown in [8], the algebra $U'_q(\mathfrak{so}_n)$ admits a q -analogue of the Gel'fand–Tsetlin formalism for construction of finite-dimensional irreducible representations. Since the algebra $U'_q(\mathfrak{so}_n)$ is not a Hopf algebra, there is no natural way to introduce the notion of tensor product of representations. But, as shown in [9 – 11], the algebra $U'_q(\mathfrak{so}_n)$ is a subalgebra in the Drinfeld–Jimbo Hopf algebra $U_q(\mathfrak{sl}_n)$. Moreover, it is possible to show that the algebra $U'_q(\mathfrak{so}_n)$ is a $U_q(\mathfrak{sl}_n)$ -comodule algebra such that the coaction coincides with the multiplication in $U_q(\mathfrak{sl}_n)$ if one embeds $U'_q(\mathfrak{so}_n)$ into $U_q(\mathfrak{sl}_n)$. This comodule structure can be used to introduce the tensor product of vector and arbitrary

representations T of $U'_q(\mathfrak{so}_n)$ (it will be denoted by T^\otimes), see [12].

We describe the decomposition of T^\otimes into irreducible subrepresentations and write down the corresponding Clebsch–Gordan coefficients in the case where T is an irreducible finite-dimensional representation of the classical or nonclassical type. The decomposition of T^\otimes in the case of classical-type representations has the same form as in the case of the Lie algebra \mathfrak{so}_n , and the corresponding Clebsch–Gordan coefficients are q -deformation of their classical analogues [13, 14].

It is well known that the Wigner–Eckart theorem for the tensor operators with respect to the Lie algebra \mathfrak{so}_n (and, especially, \mathfrak{so}_3) is very important in physics. In this paper, we give a q -analogue of such theorem in the case of vector operators.

Everywhere below, we suppose that q is not a root of unity.

2. The q -deformed Algebra $U'_q(\mathfrak{so}_n)$ and Quantum Algebra $U_q(\mathfrak{sl}_n)$

According to [3], the nonstandard q -deformation $U'_q(\mathfrak{so}_n)$ of the Lie algebra \mathfrak{so}_n is given as a complex associative algebra with $n-1$ generating elements $I_{21}, I_{32}, \dots, I_{n,n-1}$ obeying the defining relations

$$\begin{aligned} & I_{j,j-1}^2 I_{j-1,j-2} + I_{j-1,j-2} I_{j,j-1}^2 - \\ & - [2] I_{j,j-1} I_{j-1,j-2} I_{j,j-1} = -I_{j-1,j-2}, \\ & I_{j-1,j-2}^2 I_{j,j-1} + I_{j,j-1} I_{j-1,j-2}^2 - \\ & - [2] I_{j-1,j-2} I_{j,j-1} I_{j-1,j-2} = -I_{j,j-1}, \\ & [I_{i,i-1}, I_{j,j-1}] = 0 \quad \text{if } |i-j| > 1, \end{aligned} \quad (1)$$

where $q + q^{-1} \equiv [2]$, $q \in \mathbf{C}$, $q \neq 0, \pm 1$. It is useful to introduce the generators

$$I_{k,l}^\pm \equiv [I_{l+1,l}, I_{k,l+1}^\pm]_{q^{\pm 1}}, \quad k > l + 1, \quad 1 \leq k, l \leq n, \quad (2)$$

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where $[X, Y]_{q^{\pm 1}} \equiv q^{\pm 1/2}XY - q^{\mp 1/2}YX$ and $I_{k+1, k}^+ \equiv I_{k+1, k}^- \equiv I_{k+1, k}$. If $q \rightarrow 1$ ('classical' limit), the set of relations (1) reduces to those of $U(\mathfrak{so}_n)$.

The algebra $U'_q(\mathfrak{so}_n)$ can be embedded into the quantum algebra $U_q(\mathfrak{sl}_n)$, which is defined [1, 2, 15] as a complex associative algebra with the generating elements $e_i, f_i, k_i, k_i^{-1}, i = 1, 2, \dots, n - 1$, and defining relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,$$

$$k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j,$$

$$[e_i, e_j] = [f_i, f_j] = 0, \quad |i - j| > 1,$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}},$$

$$e_i^2 e_{i\pm 1} - (q + q^{-1}) e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 = 0,$$

$$f_i^2 f_{i\pm 1} - (q + q^{-1}) f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 = 0,$$

where $a_{ii} = 2, a_{i, i\pm 1} = -1$ and $a_{ij} = 0$ for $|i - j| > 1$. It is shown in [9],[10], that the elements $\tilde{I}_{i+1, i} = f_i - q^{-1} k_i e_i, i = 1, 2, \dots, n - 1$, satisfy relations (1) and define a homomorphism $U'_q(\mathfrak{so}_n) \rightarrow U_q(\mathfrak{sl}_n)$. Moreover, it is proved in [11] that this homomorphism is an embedding, that is, we may consider $U'_q(\mathfrak{so}_n)$ as a subalgebra in $U_q(\mathfrak{sl}_n)$.

The quantum algebra $U_q(\mathfrak{sl}_n)$ possesses the Hopf structure. Comultiplication on the generators of this algebra can be defined as

$$\Delta(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i,$$

$$\Delta(f_i) = f_i \otimes 1 + k_i \otimes f_i,$$

$$\Delta(k_i) = k_i \otimes k_i.$$

Therefore, we obtain the coideal property of $U'_q(\mathfrak{so}_n)$ embedded into $U_q(\mathfrak{sl}_n)$:

$$\Delta(\tilde{I}_{i+1, i}) = \tilde{I}_{i+1, i} \otimes 1 + k_i \otimes \tilde{I}_{i+1, i}.$$

Proposition 1. *The algebra $U'_q(\mathfrak{so}_n)$ is a $U_q(\mathfrak{sl}_n)$ -comodule algebra with the coaction $\phi(I_{i+1, i}) = \tilde{I}_{i+1, i} \otimes 1 + k_i \otimes I_{i+1, i}$. If one embeds $U'_q(\mathfrak{so}_n)$ into $U_q(\mathfrak{sl}_n)$, the coaction ϕ reduces to a comultiplication Δ of $U_q(\mathfrak{sl}_n)$.*

Proof. This proposition can be verified by direct calculation. \square

In particular, Proposition 1 claims that ϕ is a homomorphism from $U'_q(\mathfrak{so}_n)$ into $U_q(\mathfrak{sl}_n) \otimes U'_q(\mathfrak{so}_n)$. This comodule structure can be used to introduce the

tensor product of vector and arbitrary representations T of $U'_q(\mathfrak{so}_n)$.

Let T be a representation of $U'_q(\mathfrak{so}_n)$ on the linear space \mathcal{V} with the basis $\{v_\alpha\}$ and \mathcal{V}_1 be the n -dimensional linear space with the basis $\{v_k\}, k = 1, 2, \dots, n$, and $\mathcal{V}^\otimes \equiv \mathcal{V}_1 \otimes \mathcal{V}$.

Proposition 2. *The map T^\otimes from $U'_q(\mathfrak{so}_n)$ to $\text{End } \mathcal{V}^\otimes$ given by the formulas*

$$T^\otimes(I_{j, j-1})(v_{j-1} \otimes v_\alpha) = q v_{j-1} \otimes T(I_{j, j-1})v_\alpha - q^{1/2} v_j \otimes v_\alpha, \tag{3}$$

$$T^\otimes(I_{j, j-1})(v_j \otimes v_\alpha) = q^{-1} v_j \otimes T(I_{j, j-1})v_\alpha + q^{-1/2} v_{j-1} \otimes v_\alpha, \tag{4}$$

$$T^\otimes(I_{j, j-1})(v_k \otimes v_\alpha) = v_k \otimes T(I_{j, j-1})v_\alpha, \tag{5}$$

$j \neq k, j - 1 \neq k$

defines a representation of $U'_q(\mathfrak{so}_n)$ on the space \mathcal{V}^\otimes .

Proof. Let us define representation \mathcal{T}_1 of $U_q(\mathfrak{sl}_n)$ on the space \mathcal{V}_1 by the formulas

$$\begin{aligned} \mathcal{T}_1(e_i)v_k &= -q^{-1/2} \delta_{i+1, k} v_{k-1}, \\ \mathcal{T}_1(f_i)v_k &= -q^{1/2} \delta_{i, k} v_{k+1}, \\ \mathcal{T}_1(k_i)v_k &= q^{\delta_{i, k} - \delta_{i+1, k}} v_k. \end{aligned} \tag{6}$$

It is easy to verify that this representation is a vector representation (that is, the representation with the highest weight $(1, 0, \dots, 0)$). The action formulas (6) imply

$$\mathcal{T}_1(\tilde{I}_{i+1, i})v_k = -q^{1/2} \delta_{i, k} v_{k+1} + q^{-1/2} \delta_{i+1, k} v_{k-1}. \tag{7}$$

This representation of $U'_q(\mathfrak{so}_n)$ is equivalent to the classical type representation $T_{\mathbf{m}_n}$ with $\mathbf{m}_n = (1, 0, \dots, 0)$, that is, the vector representation (see next section). Hence, similarly to the classical case, the restriction of the vector representation of $U_q(\mathfrak{sl}_n)$ onto $U'_q(\mathfrak{so}_n)$ is the vector representation of $U'_q(\mathfrak{so}_n)$. This proposition immediately follows from Proposition 1 and formula (7), if one takes $T^\otimes = (\mathcal{T}_1 \otimes T) \circ \phi$. \square

In the case where T is the trivial representation of $U'_q(\mathfrak{so}_n)$ given by formulas $T(a) = 0, a \in U'_q(\mathfrak{so}_n), a \neq 1$, Proposition 2 gives us a representation on the space $\mathcal{V}_1 \sim \mathcal{V}^\otimes$. We denote this representation by T_1 .

$$T_1(I_{j, j-1})v_k = -q^{1/2} \delta_{k, j-1} v_j + q^{-1/2} \delta_{k, j} v_{j-1}.$$

The representations T_1 and $T_{\mathbf{m}_n}, \mathbf{m}_n = (1, 0, \dots, 0)$ (see next section), are equivalent.

In the limit $q \rightarrow 1$, Proposition 2 defines the representation which is the tensor product of the vector and some arbitrary representation of the Lie algebra \mathfrak{so}_n . On the base of these two arguments, we shall also use the notion $T^\otimes \equiv T_1 \otimes T$.

3. Finite-dimensional Classical Type Representations of $U'_q(\mathfrak{so}_n)$

In this section, we describe (in the framework of the Gel'fand–Tsetlin formalism) irreducible finite-dimensional representation of the algebra $U'_q(\mathfrak{so}_n)$, which are q -deformations of the finite-dimensional irreducible representations of the Lie algebra \mathfrak{so}_n . They are given by sets \mathbf{m}_n consisting of $\lfloor n/2 \rfloor$ numbers $m_{1,n}, m_{2,n}, \dots, m_{\lfloor n/2 \rfloor, n}$ (here $\lfloor n/2 \rfloor$ denotes the integral part of $n/2$) which are all integral or all half-integral and satisfy the dominance conditions

$$\begin{aligned} m_{1,2p+1} \geq m_{2,2p+1} \geq \dots \geq m_{p,2p+1} \geq 0, \\ m_{1,2p} \geq m_{2,2p} \geq \dots \geq m_{p-1,2p} \geq |m_{p,2p}| \end{aligned} \quad (8)$$

for $n = 2p + 1$ and $n = 2p$, respectively. These representations are denoted by $T_{\mathbf{m}_n}$. For a basis in a representation space $\mathcal{V}_{\mathbf{m}_n}$, we take the q -analogue of the Gel'fand–Tsetlin basis which is obtained by successive reduction of the representation $T_{\mathbf{m}_n}$ to the subalgebras $U'_q(\mathfrak{so}_{n-1}), U'_q(\mathfrak{so}_{n-2}), \dots, U'_q(\mathfrak{so}_3), U'_q(\mathfrak{so}_2) \equiv U(\mathfrak{so}_2)$. As in the classical case, its elements are labelled by the Gel'fand–Tsetlin tableaux

$$\begin{aligned} \{\xi_n\} &\equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\} \equiv \dots \\ &\equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \dots, \mathbf{m}_2\}, \end{aligned} \quad (9)$$

where the components of \mathbf{m}_k and \mathbf{m}_{k-1} satisfy the “betweenness” conditions

$$\begin{aligned} m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq \dots \\ \geq m_{p,2p+1} \geq m_{p,2p} \geq -m_{p,2p+1}, \\ m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq \dots \\ \geq m_{p-1,2p-1} \geq |m_{p,2p}|. \end{aligned}$$

The basis element defined by the tableau $\{\xi_n\}$ is denoted as $|\xi_n\rangle$. We suppose that the representation space $\mathcal{V}_{\mathbf{m}_n}$ is a Hilbert space and vectors $|\xi_n\rangle$ are orthonormal. It is convenient to introduce the so-called l -coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \quad l_{j,2p} = m_{j,2p} + p - j \quad (10)$$

for the numbers $m_{i,k}$. The operator $T_{\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\mathbf{m}_n}$ of $U'_q(\mathfrak{so}_n)$ acts upon Gel'fand–Tsetlin basis elements, labelled by (9), as

$$\begin{aligned} T_{\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle &= \sum_{j=1}^p A_{2p}^j(\xi_n)|(\xi_n)_{2p}^{+j}\rangle - \\ &- \sum_{j=1}^p A_{2p}^j((\xi_n)_{2p}^{-j})|(\xi_n)_{2p}^{-j}\rangle \end{aligned} \quad (11)$$

and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$\begin{aligned} T_{\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle &= \sum_{j=1}^{p-1} B_{2p-1}^j(\xi_n)|(\xi_n)_{2p-1}^{+j}\rangle - \\ &- \sum_{j=1}^{p-1} B_{2p-1}^j((\xi_n)_{2p-1}^{-j})|(\xi_n)_{2p-1}^{-j}\rangle + \\ &+ i C_{2p-1}(\xi_n)|\xi_n\rangle, \end{aligned} \quad (12)$$

$$T_{\mathbf{m}_n}(I_{21})|\xi_n\rangle = i[l_{12}]|\xi_n\rangle.$$

In these formulas, $(\xi_n)_k^{\pm j}$ means tableau (9) in which j -th component $m_{j,k}$ in \mathbf{m}_k is replaced by $m_{j,k} \pm 1$. The coefficients $A_{2p}^j, B_{2p-1}^j, C_{2p-1}$ in (11) and (12) are given by the expressions

$$A_{2p}^j(\xi_n) = \left(\frac{[l_{j,2p}][l_{j,2p} + 1]}{[2l_{j,2p}][2l_{j,2p} + 2]} \right)^{\frac{1}{2}} \hat{A}_{2p}^j, \quad (13)$$

$$B_{2p-1}^j(\xi_n) = \frac{\hat{B}_{2p-1}^j(\xi_n)}{[l_{j,2p-1}][2l_{j,2p-1} + 1][2l_{j,2p-1} - 1]}^{\frac{1}{2}}, \quad (14)$$

$$\begin{aligned} \hat{A}_{2p}^j &= \left(\frac{\prod_{i=1}^p [l_{i,2p+1} + l_{j,2p}][l_{i,2p+1} - l_{j,2p} - 1]}{\prod_{i \neq j}^p [l_{i,2p} + l_{j,2p}][l_{i,2p} - l_{j,2p}]} \right. \\ &\times \left. \frac{\prod_{i=1}^{p-1} [l_{i,2p-1} + l_{j,2p}][l_{i,2p-1} - l_{j,2p} - 1]}{\prod_{i \neq j}^p [l_{i,2p} + l_{j,2p} + 1][l_{i,2p} - l_{j,2p} - 1]} \right)^{\frac{1}{2}} \end{aligned} \quad (15)$$

and

$$\begin{aligned} \hat{B}_{2p-1}^j(\xi_n) &= \left(\frac{\prod_{i=1}^p [l_{i,2p} + l_{j,2p-1}][l_{i,2p} - l_{j,2p-1}]}{\prod_{i \neq j}^{p-1} [l_{i,2p-1} + l_{j,2p-1}][l_{i,2p-1} - l_{j,2p-1}]} \right. \\ &\times \left. \frac{\prod_{i=1}^{p-1} [l_{i,2p-2} + l_{j,2p-1}][l_{i,2p-2} - l_{j,2p-1}]}{\prod_{i \neq j}^{p-1} [l_{i,2p-1} + l_{j,2p-1} - 1][l_{i,2p-1} - l_{j,2p-1} - 1]} \right)^{\frac{1}{2}}, \end{aligned} \quad (16)$$

$$C_{2p-1}(\xi_n) = \frac{\prod_{i=1}^p [l_{i,2p}][l_{i,2p-2}]}{\prod_{i=1}^{p-1} [l_{i,2p-1}][l_{i,2p-1} - 1]}, \quad (17)$$

where numbers in square brackets mean q -numbers defined by $[a] := (q^a - q^{-a})/(q - q^{-1})$.

4. Finite Dimensional Nonclassical Type Representations of $U'_q(\mathfrak{so}_n)$

The representations of the previous section are called representations of the classical type, because, at $q \rightarrow 1$, the operators $T_{\mathbf{m}_n}(I_{j,j-1})$ turn into the corresponding operators $T_{\mathbf{m}_n}(I_{j,j-1})$ for irreducible finite-dimensional representations with the highest weights \mathbf{m}_n of the Lie algebra \mathfrak{so}_n .

The algebra $U'_q(\mathfrak{so}_n)$ also has irreducible finite-dimensional representations T of nonclassical type, that is, such that the operators $T(I_{j,j-1})$ have no classical

limit $q \rightarrow 1$. They are given (see [16]) by sets $\epsilon := (\epsilon_2, \epsilon_3, \dots, \epsilon_n)$, $\epsilon_i = \pm 1$, and by sets \mathbf{m}_n consisting of $[n/2]$ **half-integral** numbers $m_{1,n}, m_{2,n}, \dots, m_{[n/2],n}$ (here $[n/2]$ denotes the integral part of $n/2$) that satisfy the dominance conditions

$$m_{1,n} \geq m_{2,n} \geq \dots \geq m_{[n/2],n} \geq 1/2. \tag{18}$$

These representations are denoted by $T_{\epsilon, \mathbf{m}_n}$.

For a basis in the representation space $\tilde{\mathcal{V}}_{\mathbf{m}_n}$, we use the analogue of the basis of the previous section. Its elements are labeled by the tableaux

$$\begin{aligned} \{\xi_n\} &\equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\} \equiv \dots \\ &\equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \dots, \mathbf{m}_2\}, \end{aligned} \tag{19}$$

where the components of \mathbf{m}_k and \mathbf{m}_{k-1} satisfy the "betweenness" conditions

$$\begin{aligned} m_{1,2p+1} &\geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq \dots \\ &\geq m_{p,2p+1} \geq m_{p,2p} \geq 1/2, \\ m_{1,2p} &\geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq \dots \\ &\geq m_{p-1,2p-1} \geq m_{p,2p}. \end{aligned}$$

The basis element defined by the tableau $\{\xi_n\}$ is denoted as $|\xi_n\rangle$. We suppose that the representation space $\tilde{\mathcal{V}}_{\mathbf{m}_n}$ is a Hilbert space and vectors $|\xi_n\rangle$ are orthonormal. It is convenient to introduce the l -coordinates as in (10).

The operator $T_{\epsilon, \mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\epsilon, \mathbf{m}_n}$ of $U'_q(\mathfrak{so}_n)$ acts upon basis elements, labelled by (19), by the formula

$$\begin{aligned} T_{\epsilon, \mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle &= \delta_{m_{p,2p}, 1/2} \frac{\epsilon_{2p+1}}{q^{1/2} - q^{-1/2}} D_{2p}(\xi_n) |\xi_n\rangle \\ &+ \sum_{j=1}^p \tilde{A}_{2p}^j(\xi_n) |(\xi_n)_{2p}^{+j}\rangle - \sum_{j=1}^p \tilde{A}_{2p}^j(\xi_n) |(\xi_n)_{2p}^{-j}\rangle, \end{aligned} \tag{20}$$

where the summation in the last sum must be from 1 to $p-1$ if $m_{p,2p} = 1/2$, and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$\begin{aligned} T_{\epsilon, \mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle &= \sum_{j=1}^{p-1} \tilde{B}_{2p-1}^j(\xi_n) |(\xi_n)_{2p-1}^{+j}\rangle - \\ &- \sum_{j=1}^{p-1} \tilde{B}_{2p-1}^j(\xi_n) |(\xi_n)_{2p-1}^{-j}\rangle + \epsilon_{2p} \tilde{C}_{2p-1}(\xi_n) |\xi_n\rangle, \end{aligned} \tag{21}$$

$$T_{\epsilon, \mathbf{m}_n}(I_{21})|\xi_n\rangle = \epsilon_2 [l_{12}]_+ |\xi_n\rangle,$$

where $[a]_+ := (q^a + q^{-a})/(q - q^{-1})$. In these formulas, $(\xi_n)_k^{\pm j}$ means tableau (19), in which the j -th component

$m_{j,k}$ in \mathbf{m}_k is replaced by $m_{j,k} \pm 1$. Matrix elements \tilde{A}_{2p}^j and \tilde{B}_{2p-1}^j are defined using formulas (15) and (16):

$$\tilde{A}_{2p}^j(\xi_n) = \frac{\hat{A}_{2p}^j(\xi_n)}{((q^{l_{j,2p}} - q^{-l_{j,2p}})(q^{l_{j,2p+1}} - q^{-l_{j,2p+1}}))^{\frac{1}{2}}},$$

$$\tilde{B}_{2p-1}^j(\xi_n) = \frac{\hat{B}_{2p-1}^j(\xi_n)}{[l_{j,2p-1}]_+ ([2l_{j,2p-1} + 1][2l_{j,2p-1} - 1])^{\frac{1}{2}}},$$

$$\tilde{C}_{2p-1}(\xi_n) = \frac{\prod_{s=1}^p [l_{s,2p}]_+ \prod_{s=1}^{p-1} [l_{s,2p-2}]_+}{\prod_{s=1}^{p-1} [l_{s,2p-1}]_+ [l_{s,2p-1} - 1]_+},$$

$$D_{2p}(\xi_n) = \frac{\prod_{i=1}^p [l_{i,2p+1} - \frac{1}{2}] \prod_{i=1}^{p-1} [l_{i,2p-1} - \frac{1}{2}]}{\prod_{i=1}^{p-1} [l_{i,2p} + \frac{1}{2}] [l_{i,2p} - \frac{1}{2}]}.$$

5. Decomposition of Representations $T_1 \otimes T_{\mathbf{m}_3}$ of the Algebra $U'_q(\mathfrak{so}_3)$

In this and in the next sections, we consider the decomposition of representations $T^\otimes \equiv T_1 \otimes T_{\mathbf{m}_n}$ into irreducible constituents of the algebra $U'_q(\mathfrak{so}_n)$. In this section, we restrict ourselves to the case $n = 2, 3$.

First, we consider the case of the algebra $U'_q(\mathfrak{so}_2) \equiv U(\mathfrak{so}_2)$. This algebra has representations T_m , $m \equiv m_{12}$, $m \in \frac{1}{2}\mathbf{Z}$, of the *classical type* acting on one-dimensional spaces with basis vectors $|m\rangle$, and $T_m(I_{21})|m\rangle = i[m]|m\rangle$. Then

$$T^\otimes(I_{21})(v_1 \otimes |m\rangle) = iq[m]v_1 \otimes |m\rangle - q^{1/2}v_2 \otimes |m\rangle,$$

$$T^\otimes(I_{21})(v_2 \otimes |m\rangle) = iq^{-1}[m]v_2 \otimes |m\rangle + q^{-1/2}v_1 \otimes |m\rangle.$$

This representation is reducible. We introduce the vectors

$$v_\pm^{(m)} = \mp iq^{-1/2 \pm m} v_1 + v_2. \tag{22}$$

Then the vectors $|m \pm 1\rangle^\otimes := v_\pm^{(m)} \otimes |m\rangle$ are the eigenvectors of $T^\otimes(I_{21})$: $T^\otimes(I_{21})|m \pm 1\rangle^\otimes = i[m \pm 1]|m \pm 1\rangle^\otimes$. This fact can be easily verified by direct calculation using the definition of q -numbers. Thus, we have decomposition $T^\otimes \equiv T_1 \otimes T_m = T_{m+1} \oplus T_{m-1}$.

Now, we consider the case of the algebra $U'_q(\mathfrak{so}_3)$. This algebra has representations T_l , $\mathbf{m}_3 \equiv (m_{13}) \equiv (l)$, $l \in \{0, 1/2, 1, 3/2, \dots\}$, of the *classical type* acting on the spaces \mathcal{V}_l with the basis vectors $|l, m\rangle$, ($m \equiv m_{12}$), $m = -l, -l+1, \dots, l$:

$$T_l(I_{21})|l, m\rangle = i[m]|l, m\rangle,$$

$$T_l(I_{32})|l, m\rangle = A_{l,m}|l, m+1\rangle - A_{l,m-1}|l, m-1\rangle,$$

where $A_{l,m} = d_m([l - m][l + m + 1])^{1/2}$, $d_m = ([m][m + 1]/([2m][2m + 2]))^{1/2}$. Let us consider the vectors

$$|l', m\rangle^{\otimes} := \alpha_{l,m}^{(l')} v_+^{(m-1)} \otimes |l, m - 1\rangle + \beta_{l,m}^{(l')} v_3 \otimes |l, m\rangle + \gamma_{l,m}^{(l')} v_-^{(m+1)} \otimes |l, m + 1\rangle, \quad (23)$$

where $m = -l', -l' + 1, \dots, l'$, and

$$l' = l + 1, l, l - 1 \quad \text{if } l \geq 1;$$

$$l' = 3/2, 1/2 \quad \text{if } l = 1/2; \quad l' = 1 \quad \text{if } l = 0.$$

The vectors $v_{\pm}^{(m)}$ in (23) are defined in (22) and

$$\alpha_{l,m}^{(l+1)} = q^{l-m+1/2} d_{m-1}([l + m][l + m + 1])^{1/2},$$

$$\beta_{l,m}^{(l+1)} = ([l - m + 1][l + m + 1])^{1/2},$$

$$\gamma_{l,m}^{(l+1)} = -q^{l+m+1/2} d_m([l - m][l - m + 1])^{1/2},$$

$$\alpha_{l,m}^{(l)} = -q^{-m-1/2} d_{m-1}([l + m][l - m + 1])^{1/2},$$

$$\beta_{l,m}^{(l)} = [m],$$

$$\gamma_{l,m}^{(l)} = -q^{m-1/2} d_m([l - m][l + m + 1])^{1/2},$$

$$\alpha_{l,m}^{(l-1)} = -q^{-l-m-1/2} d_{m-1}([l - m][l - m + 1])^{1/2},$$

$$\beta_{l,m}^{(l-1)} = ([l - m][l + m])^{1/2},$$

$$\gamma_{l,m}^{(l-1)} = q^{-l+m-1/2} d_m([l + m][l + m + 1])^{1/2}.$$

In the case of $U'_q(\mathfrak{so}_2)$, it is easy to see that $T^{\otimes}(I_{21})|l', m\rangle^{\otimes} = i[m]|l', m\rangle^{\otimes}$. One can show by direct calculation that $T^{\otimes}(I_{32})|l', m\rangle^{\otimes} = A_{l',m}|l', m + 1\rangle^{\otimes} - A_{l',m-1}|l', m - 1\rangle^{\otimes}$. It means that the vectors $|l', m\rangle^{\otimes}$ at fixed l' span a subspace in \mathcal{V}^{\otimes} , which is invariant and irreducible under the action of $T^{\otimes}(a)$, $a \in U'_q(\mathfrak{so}_3)$. The corresponding subrepresentation is equivalent to $T_{l'}$. Comparing the dimensions of $T_{l'}$ with the dimension of T^{\otimes} , we conclude that $T^{\otimes} = T_{l+1} \oplus T_l \oplus T_{l-1}$ if $l \geq 1$; $T^{\otimes} = T_{3/2} \oplus T_{1/2}$, if $l = 1/2$; $T^{\otimes} = T_1$ if $l = 0$. Let us recall that $T_l \equiv T_{\mathbf{m}_3}$, $m_{13} \equiv l$. The numbers $\alpha_{l,m}^{(l')}$, $\beta_{l,m}^{(l')}$ and $\gamma_{l,m}^{(l')}$ are the Clebsch–Gordan coefficients of these decompositions.

6. Decomposition of $T_1 \otimes T_{\mathbf{m}_n}$ of the Algebra $U'_q(\mathfrak{so}_n)$, $n \geq 4$

In this section, we consider the decomposition of the representations $T^{\otimes} \equiv T_1 \otimes T_{\mathbf{m}_n}$ of the algebra $U'_q(\mathfrak{so}_n)$, $n \geq 4$, into irreducible constituents. All the results of this section are obtained in [12]. As shown there, this decomposition has the form

$$T^{\otimes} = \bigoplus_{\mathbf{m}'_n \in \mathcal{S}(\mathbf{m}_n)} T_{\mathbf{m}'_n}, \quad (24)$$

where

$$\mathcal{S}(\mathbf{m}_{2p+1}) = \bigcup_{j=1}^p \{\mathbf{m}_{2p+1}^{+j}\} \cup \bigcup_{j=1}^p \{\mathbf{m}_{2p+1}^{-j}\} \cup \{\mathbf{m}_{2p+1}\}, \quad (25)$$

$$\mathcal{S}(\mathbf{m}_{2p}) = \bigcup_{j=1}^p \{\mathbf{m}_{2p}^{+j}\} \cup \bigcup_{j=1}^p \{\mathbf{m}_{2p}^{-j}\}. \quad (26)$$

By $\mathbf{m}_n^{\pm j}$, we mean the set \mathbf{m}_n with $m_{j,n}$ replaced by $m_{j,n} \pm 1$, respectively. If some $\mathbf{m}_n^{\pm j}$ is not dominant (8), then the corresponding $\mathbf{m}_n^{\pm j}$ must be omitted. If $m_{p,2p+1} = 0$, then \mathbf{m}_{2p+1} on right-hand side of (25) also must be omitted. Decomposition (24) of the representation T^{\otimes} corresponds to the decomposition of the carrier space:

$$\mathcal{V}^{\otimes} \equiv \mathcal{V}_1 \otimes \mathcal{V}_{\mathbf{m}_n} = \bigoplus_{\mathbf{m}'_n \in \mathcal{S}(\mathbf{m}_n)} \mathcal{V}_{\mathbf{m}'_n}. \quad (27)$$

In order to give this decomposition in explicit form, we change the basis $\{v_k \otimes |\xi_n\rangle\}$, $k = 1, 2, \dots, n$, in \mathcal{V}^{\otimes} to $\{v_k \otimes |\xi_n\rangle\}$, $k = +, -, 3, \dots, n$, by replacing (for every fixed $\{\xi_n\} = \{\mathbf{m}_n, \mathbf{m}_{n-1}, \dots, \mathbf{m}_3, \mathbf{m}_2\}$) two basis vectors $v_1 \otimes |\xi_n\rangle$ and $v_2 \otimes |\xi_n\rangle$ by $v_+^{(m_{12})} \otimes |\xi_n\rangle$ and $v_-^{(m_{12})} \otimes |\xi_n\rangle$ (see (22)). From now on, we shall omit the index (m_{12}) in the notion of the basis vectors $v_{\pm}^{(m_{12})} \otimes |\xi_n\rangle$, supposing that it is equal to the m_{12} -component of the corresponding Gel'fand–Tsetlin tableaux $\{\xi_n\}$.

We introduce the vectors (where $\{\xi'_n\} = \{\mathbf{m}'_n, \mathbf{m}'_{n-1}, \dots, \mathbf{m}'_3, \mathbf{m}'_2\}$)

$$|\mathbf{m}'_n, \xi_{n-1}\rangle^{\otimes} := \sum_k \sum_{|\mathbf{m}_n, \xi'_{n-1}\rangle \in \mathcal{V}_{\mathbf{m}_n}} (k, (\mathbf{m}_n, \xi'_{n-1}) | (\mathbf{m}'_n, \xi_{n-1})) \times v_k \otimes |\mathbf{m}_n, \xi'_{n-1}\rangle \quad (28)$$

in the space \mathcal{V}^{\otimes} , where k runs over the set $+, -, 3, \dots, n$, and the coefficients $(k, (\mathbf{m}_n, \xi'_{n-1}) | (\mathbf{m}'_n, \xi_{n-1}))$ are Clebsch–Gordan coefficients (CGC's). Now we define these CGC's in explicit form.

We put $(k, (\mathbf{m}_n, \xi'_{n-1}) | (\mathbf{m}'_n, \xi_{n-1})) = 0$ if one of the conditions

- 1) $\mathbf{m}'_n \notin \mathcal{S}(\mathbf{m}_n)$,
- 2) $\mathbf{m}_s \notin \mathcal{S}(\mathbf{m}'_s), s = n-1, \dots, k, k \geq 3$,
- 3) $\mathbf{m}_s \notin \mathcal{S}(\mathbf{m}'_s), s = n-1, \dots, 3, k = +, -$,
- 4) $\xi'_{k-1} \neq \xi_{k-1}, k = 3, 4, \dots, n$,
- 5) $m_{12} \neq m'_{12} + 1, k = +$,
- 6) $m_{12} \neq m'_{12} - 1, k = -$

is fulfilled. The nonzero CGC for $k = n$ are:

$$\begin{aligned} & (2p+1, (\mathbf{m}_{2p+1}, \xi_{2p}) | (\mathbf{m}_{2p+1}^{+j}, \xi_{2p})) = \\ & = \left(\prod_{r=1}^p [l_{j,2p+1} + l_{r,2p}] [l_{j,2p+1} - l_{r,2p}] \right)^{\frac{1}{2}}, \\ & (2p+1, (\mathbf{m}_{2p+1}, \xi_{2p}) | (\mathbf{m}_{2p+1}, \xi_{2p})) = \\ & = \prod_{r=1}^p [l_{r,2p}], \\ & (2p+1, (\mathbf{m}_{2p+1}, \xi_{2p}) | (\mathbf{m}_{2p+1}^{-j}, \xi_{2p})) = \\ & = \left(\prod_{r=1}^p [l_{j,2p+1} + l_{r,2p} - 1] [l_{j,2p+1} - l_{r,2p} - 1] \right)^{\frac{1}{2}}, \end{aligned} \quad (29)$$

$$\begin{aligned} & (2p, (\mathbf{m}_{2p}, \xi_{2p-1}) | (\mathbf{m}_{2p}^{+j}, \xi_{2p-1})) = \\ & = \left(\prod_{r=1}^{p-1} [l_{j,2p} + l_{r,2p-1}] [l_{j,2p} - l_{r,2p-1} + 1] \right)^{\frac{1}{2}}, \\ & (2p, (\mathbf{m}_{2p}, \xi_{2p-1}) | (\mathbf{m}_{2p}^{-j}, \xi_{2p-1})) = \\ & = \left(\prod_{r=1}^{p-1} [l_{j,2p} + l_{r,2p-1} - 1] [l_{j,2p} - l_{r,2p-1}] \right)^{\frac{1}{2}}. \end{aligned} \quad (30)$$

(They are defined up to normalization, that is, multiplication of these CGC's by some constants will not spoil the following results.)

All the other CGC's can be found from the just presented as follows:

$$\begin{aligned} & (k, \xi_n | \xi'_n) = q^{k-n} \times \\ & \times \frac{\langle \mathbf{m}_{n+1}, \xi_n | T_{\mathbf{m}_{n+1}}(I_{n+1,k}^-) | \mathbf{m}_{n+1}, \xi'_n \rangle}{\langle \mathbf{m}_{n+1}, \mathbf{m}_n, \xi_{n-1} | T_{\mathbf{m}_{n+1}}(I_{n+1,n}) | \mathbf{m}_{n+1}, \mathbf{m}'_n, \xi_{n-1} \rangle} \times \\ & \times (n, (\mathbf{m}_n, \xi_{n-1}) | (\mathbf{m}'_n, \xi_{n-1})), \end{aligned} \quad (31)$$

where the generators $I_{n+1,k}^-$ are defined in (2). If $k = +$ or $k = -$ on the left-hand side of (31), one must put $k=2$ on

right-hand side. The set \mathbf{m}_{n+1} must be chosen to give a non-zero denominator on right-hand side of (31). Note that if $(n, (\mathbf{m}_n, \xi_{n-1}) | (\mathbf{m}'_n, \xi_{n-1})) \neq 0$, one can always do such a choice, moreover, the resulting CGC will not depend on this particular choice. In case of $n=3$, we re-obtain the CGC's for the algebra $U'_q(\mathfrak{so}_3)$ (see Section 5).

As shown in [12], the defined CGC's have the *factorization* property. This fact (in complete analogy with the classical case, see [13, 14]) gives a possibility to present arbitrary CGC for the algebra $U'_q(\mathfrak{so}_n)$ as a product of *scalar factors*.

Theorem 1. *The formulas for the action of the operators $T^\otimes(I_{k+1,k})$, $k = 1, 2, \dots, n-1$, on the vectors $|\mathbf{m}'_n, \xi_{n-1}\rangle^\otimes$ defined by (28) with CGC's defined by (29)–(31), coincide with the corresponding formulas (11)–(12) for the action of the operators $T_{\mathbf{m}'_n}(I_{k+1,k})$ on the GT basis vectors $|\mathbf{m}'_n, \xi_{n-1}\rangle$. We have decomposition (24).*

7. Decomposition of Representations

$T_1 \otimes T_{\epsilon, \mathfrak{m}_3}$ of the Algebra $U'_q(\mathfrak{so}_3)$

In this and in the next section, we consider the decomposition of representations $T^\otimes \equiv T_1 \otimes T_{\epsilon, \mathfrak{m}_n}$ into irreducible constituents of the algebra $U'_q(\mathfrak{so}_n)$. In this section, we restrict ourselves to the case $n = 2, 3$.

First, we consider the case of the algebra $U'_q(\mathfrak{so}_2) \equiv U(\mathfrak{so}_2)$. This algebra has representations $T_{\epsilon_2, m}$, $\epsilon_2 = \pm 1$, $m \equiv m_{12}$, $m \in \{1/2, 3/2, \dots\}$, acting on one-dimensional spaces with basis vectors $|m\rangle$, and $T_{\epsilon_2, m}(I_{21})|m\rangle = \epsilon_2[m]_{\pm}|m\rangle$. Then the representation $T^\otimes \equiv T_1 \otimes T_{\epsilon_2, m}$ is two-dimensional and reducible. We introduce the vectors

$$v_{\pm}^{(\epsilon_2, m)} = -\epsilon_2 q^{-1/2 \pm m} v_1 + v_2. \quad (32)$$

Then the vectors $|m \pm 1\rangle^\otimes := v_{\pm}^{(\epsilon_2, m)} \otimes |m\rangle$ are eigenvectors of $T^\otimes(I_{21})$: $T^\otimes(I_{21})|m \pm 1\rangle^\otimes = \epsilon_2[m \pm 1]_{\pm}|m \pm 1\rangle^\otimes$. This fact can be easily verified by direct calculation using the definition of q -numbers. Thus, we have decomposition $T^\otimes \equiv T_1 \otimes T_{\epsilon, m} = T_{\epsilon_2, m+1} \oplus T_{\epsilon_2, m-1}$ if $m \geq 3/2$, and $T^\otimes \equiv T_1 \otimes T_{\epsilon_2, 1/2} = T_{\epsilon_2, 3/2} \oplus T_{\epsilon_2, 1/2}$ if $m = 1/2$.

Now, we consider the case of the algebra $U'_q(\mathfrak{so}_3)$. This algebra has four classes of representations of the *nonclassical type* $T_{\epsilon, l}$, $\epsilon = \{\epsilon_2, \epsilon_3\}$, $\epsilon_i \in \{\pm 1\}$, $\mathfrak{m}_3 \equiv (m_{13}) \equiv (l)$, $l \in \{1/2, 3/2, 5/2, \dots\}$, acting on the spaces \mathcal{V}_l with the basis vectors $|l, m\rangle$, ($m \equiv m_{12}$), $m = 1/2, 3/2, \dots, l$:

$$T_{\epsilon, l}(I_{21})|l, m\rangle = \epsilon_2[m]_{\pm}|l, m\rangle,$$

$$T_{\epsilon,l}(I_{32})|l,m\rangle = \tilde{A}_{l,m}|l,m+1\rangle - \tilde{A}_{l,m-1}|l,m-1\rangle$$

if $m \geq 3/2$,

$$T_{\epsilon,l}(I_{32})|l,1/2\rangle = \tilde{A}_{l,1/2}|l,3/2\rangle + \epsilon_3[1/2]_+[l+1/2]|l,1/2\rangle,$$

where $\tilde{A}_{l,m} = \tilde{d}_m([l-m][l+m+1])^{1/2}$, $\tilde{d}_m = ((q^m - q^{-m})(q^{m+1} - q^{-m-1}))^{-1/2}$. Let us consider the vectors

$$\begin{aligned} |l',m\rangle^{\otimes} &:= \tilde{\alpha}_{l,m}^{(l')} v_+^{(\epsilon_2,m-1)} \otimes |l,m-1\rangle + \\ &+ \tilde{\beta}_{l,m}^{(l')} v_3 \otimes |l,m\rangle + \tilde{\gamma}_{l,m}^{(l')} v_-^{(\epsilon_2,m+1)} \otimes |l,m+1\rangle, \end{aligned} \quad (33)$$

where $m = 3/2, 5/2, \dots, l'$, and

$$l' = l+1, l, l-1 \quad \text{if } l \geq 3/2; \quad l' = 3/2, 1/2 \quad \text{if } l = 1/2.$$

If $m = 1/2$, we should replace $|l,-1/2\rangle$ by $|l,1/2\rangle$ on right-hand side of (33).

The vectors $v_{\pm}^{(\epsilon_2,m)}$ in (33) are defined in (32) and

$$\tilde{\alpha}_{l,m}^{(l+1)} = q^{l-m+1/2} \tilde{d}_{m-1}([l+m][l+m+1])^{1/2}, \quad m \neq \frac{1}{2},$$

$$\tilde{\beta}_{l,m}^{(l+1)} = ([l-m+1][l+m+1])^{1/2},$$

$$\tilde{\gamma}_{l,m}^{(l+1)} = -q^{l+m+1/2} \tilde{d}_m([l-m][l-m+1])^{1/2},$$

$$\tilde{\alpha}_{l,m}^{(l)} = q^{-m-1/2} \tilde{d}_{m-1}([l+m][l-m+1])^{1/2}, \quad m \neq \frac{1}{2},$$

$$\tilde{\beta}_{l,m}^{(l)} = [m]_+,$$

$$\tilde{\gamma}_{l,m}^{(l)} = -q^{m-1/2} \tilde{d}_m([l-m][l+m+1])^{1/2},$$

$$\tilde{\alpha}_{l,m}^{(l-1)} = -q^{-l-m-1/2} \tilde{d}_{m-1}([l-m][l-m+1])^{1/2}, \quad m \neq \frac{1}{2},$$

$$\tilde{\beta}_{l,m}^{(l-1)} = ([l-m][l+m])^{1/2},$$

$$\tilde{\gamma}_{l,m}^{(l-1)} = q^{-l+m-1/2} \tilde{d}_m([l+m][l+m+1])^{1/2},$$

$$\tilde{\alpha}_{l,1/2}^{(l+1)} = -q^l [1/2]_+ \epsilon_3 ([l+1/2][l+3/2])^{1/2},$$

$$\tilde{\alpha}_{l,1/2}^{(l+1)} = -q^{-1} [1/2]_+ \epsilon_3 [l+1/2],$$

$$\tilde{\alpha}_{l,1/2}^{(l-1)} = q^{-l-1} [1/2]_+ \epsilon_3 ([l-1/2][l+1/2])^{1/2}.$$

From the case of $U'_q(\mathfrak{so}_2)$, it is easy to see that $T^{\otimes}(I_{21})|l',m\rangle^{\otimes} = \epsilon_2[m]_+|l',m\rangle^{\otimes}$. One can show by direct calculation that the operator $T^{\otimes}(I_{32})$ acts on the set of vectors $|l',m\rangle^{\otimes}$ at some fixed l' as the operator $T_{\epsilon,l'}(I_{32})$ acts on the Gel'fand–Tsetlin basis

vectors $|l',m\rangle$. It means that the vectors $|l',m\rangle^{\otimes}$ at fixed l' span a subspace in \mathcal{V}^{\otimes} , which is invariant and irreducible under the action of $T^{\otimes}(a)$, $a \in U'_q(\mathfrak{so}_3)$. The corresponding subrepresentation is equivalent to $T_{\epsilon,l'}$. Comparing the dimensions of $T_{\epsilon,l'}$ with the dimension of T^{\otimes} , we conclude that $T^{\otimes} = T_{\epsilon,l+1} \oplus T_{\epsilon,l} \oplus T_{\epsilon,l-1}$, if $l \geq 3/2$; $T^{\otimes} = T_{\epsilon,3/2} \oplus T_{\epsilon,1/2}$, if $l = 1/2$. Let us recall that $T_{\epsilon,l} \equiv T_{\epsilon,\mathbf{m}_3}$, $m_{13} \equiv l$. The numbers $\tilde{\alpha}_{l,m}^{(l')}$, $\tilde{\beta}_{l,m}^{(l')}$ and $\tilde{\gamma}_{l,m}^{(l')}$ are the Clebsch–Gordan coefficients of these decompositions.

8. Decomposition of $T_1 \otimes T_{\epsilon,\mathbf{m}_n}$ of the Algebra $U'_q(\mathfrak{so}_n)$, $n \geq 4$

In this section, we describe the decomposition of the representations $T^{\otimes} \equiv T_1 \otimes T_{\epsilon,\mathbf{m}_n}$ of the algebra $U'_q(\mathfrak{so}_n)$, $n \geq 4$, into irreducible constituents. This decomposition has the form

$$T^{\otimes} = \bigoplus_{\mathbf{m}'_n \in \mathcal{S}(\mathbf{m}_n)} T_{\epsilon,\mathbf{m}'_n}, \quad (34)$$

where

$$\mathcal{S}(\mathbf{m}_{2p+1}) = \bigcup_{j=1}^p \{\mathbf{m}_{2p+1}^{+j}\} \cup \bigcup_{j=1}^p \{\mathbf{m}_{2p+1}^{-j}\} \cup \{\mathbf{m}_{2p+1}\}, \quad (35)$$

$$\mathcal{S}(\mathbf{m}_{2p}) = \bigcup_{j=1}^p \{\mathbf{m}_{2p}^{+j}\} \cup \bigcup_{j=1}^p \{\mathbf{m}_{2p}^{-j}\}. \quad (36)$$

By $\mathbf{m}_n^{\pm j}$, we mean the set \mathbf{m}_n with $m_{j,n}$ replaced by $m_{j,n} \pm 1$, respectively. If $m_{p,2p} = 1/2$, the element \mathbf{m}_{2p}^{-p} on right-hand side of (36) must be replaced by \mathbf{m}_{2p} . If some $\mathbf{m}_n^{\pm j}$ is not dominant (18), then the corresponding $\mathbf{m}_n^{\pm j}$ must be omitted; in particular, if $m_{p,2p+1} = 1/2$, the element \mathbf{m}_{2p+1}^{-p} must be omitted. Note, that the representation $T_1 \otimes T_{\epsilon,\mathbf{m}_n}$ decomposes into irreducible nonclassical type representations with the same set $\epsilon = (\epsilon_2, \epsilon_3, \dots)$. Decomposition (34) of the representation T^{\otimes} , corresponds to the decomposition of carrier space:

$$\mathcal{V}^{\otimes} \equiv \mathcal{V}_1 \otimes \tilde{\mathcal{V}}_{\epsilon,\mathbf{m}_n} = \bigoplus_{\mathbf{m}'_n \in \mathcal{S}(\mathbf{m}_n)} \tilde{\mathcal{V}}_{\epsilon,\mathbf{m}'_n}. \quad (37)$$

In order to give this decomposition in explicit form, we change the basis $\{v_k \otimes |\xi_n\rangle\}$, $k = 1, 2, \dots, n$, in \mathcal{V}^{\otimes} to $\{v_k \otimes |\xi_n\rangle\}$, $k = +, -, 3, \dots, n$, by replacing (for every fixed $\{\xi_n\} = \{\mathbf{m}_n, \mathbf{m}_{n-1}, \dots, \mathbf{m}_3, \mathbf{m}_2\}$) two basis vectors $v_1 \otimes |\xi_n\rangle$ and $v_2 \otimes |\xi_n\rangle$ by $v_+^{(\epsilon_2, m_{12})} \otimes |\xi_n\rangle$ and $v_-^{(\epsilon_2, m_{12})} \otimes |\xi_n\rangle$ (see (32)). From now on, we shall omit the index (ϵ_2, m_{12}) in the notion of the basis

vectors $v_{\pm}^{(\epsilon_2, m_{12})} \otimes |\xi_n\rangle$, supposing that it contains the m_{12} -component of the corresponding Gel'fand–Tsetlin tableaux $\{\xi_n\}$.

We introduce the vectors (where $\{\xi'_n\} = \{\mathbf{m}'_n, \mathbf{m}'_{n-1}, \dots, \mathbf{m}'_3, \mathbf{m}'_2\}$)

$$\begin{aligned} & |\mathbf{m}'_n, \xi_{n-1}\rangle^{\otimes} := \\ & := \sum_k \sum_{|\mathbf{m}_n, \xi'_{n-1}\rangle \in \mathcal{V}_{\mathbf{m}_n}} (k, (\mathbf{m}_n, \xi'_{n-1}) | (\mathbf{m}'_n, \xi_{n-1}); \epsilon) \times \\ & \times v_k \otimes |\mathbf{m}_n, \xi'_{n-1}\rangle \end{aligned} \quad (38)$$

in the space \mathcal{V}^{\otimes} , where k runs over the set $+, -, 3, \dots, n$, and $(k, (\mathbf{m}_n, \xi'_{n-1}) | (\mathbf{m}'_n, \xi_{n-1}); \epsilon)$ are Clebsch–Gordan coefficients (CGC's). Now we define these CGC's in explicit form.

We put $(k, (\mathbf{m}_n, \xi'_{n-1}) | (\mathbf{m}'_n, \xi_{n-1}); \epsilon) = 0$ if one of the conditions

- 1) $\mathbf{m}'_n \notin \mathcal{S}(\mathbf{m}_n)$,
- 2) $\mathbf{m}'_s \notin \mathcal{S}(\mathbf{m}'_s)$, $s = n-1, \dots, k$, $k \geq 3$,
- 3) $\mathbf{m}'_s \notin \mathcal{S}(\mathbf{m}'_s)$, $s = n-1, \dots, 3$, $k = +, -$,
- 4) $\xi'_{k-1} \neq \xi_{k-1}$, $k = 3, 4, \dots, n$,
- 5) $m_{12} \neq m'_{12} + 1$, $k = +$,
- 6) $m_{12} \neq m'_{12} - 1$, $k = -$, $m'_{12} \geq \frac{3}{2}$,
- 6') $m_{12} \neq m'_{12}$, $k = -$, $m'_{12} = \frac{1}{2}$.

is fulfilled. The nonzero CGC for $k = n$ are:

$$\begin{aligned} & (2p+1, (\mathbf{m}_{2p+1}, \xi_{2p}) | (\mathbf{m}_{2p+1}^+, \xi_{2p}); \epsilon) = \\ & = \left(\prod_{r=1}^p [l_{j,2p+1} + l_{r,2p}] [l_{j,2p+1} - l_{r,2p}] \right)^{\frac{1}{2}}, \\ & (2p+1, (\mathbf{m}_{2p+1}, \xi_{2p}) | (\mathbf{m}_{2p+1}, \xi_{2p}); \epsilon) = \prod_{r=1}^p [l_{r,2p}]_+, \\ & (2p+1, (\mathbf{m}_{2p+1}, \xi_{2p}) | (\mathbf{m}_{2p+1}^-, \xi_{2p}); \epsilon) = \\ & = \left(\prod_{r=1}^p [l_{j,2p+1} + l_{r,2p} - 1] [l_{j,2p+1} - l_{r,2p} - 1] \right)^{\frac{1}{2}}, \quad (39) \\ & (2p, (\mathbf{m}_{2p}, \xi_{2p-1}) | (\mathbf{m}_{2p}^+, \xi_{2p-1}); \epsilon) = \\ & = \left(\prod_{r=1}^{p-1} [l_{j,2p} + l_{r,2p-1}] [l_{j,2p} - l_{r,2p-1} + 1] \right)^{\frac{1}{2}}, \\ & (2p, (\mathbf{m}_{2p}, \xi_{2p-1}) | (\mathbf{m}_{2p}^-, \xi_{2p-1}); \epsilon) = \\ & = \left(\prod_{r=1}^{p-1} [l_{j,2p} + l_{r,2p-1} - 1] [l_{j,2p} - l_{r,2p-1}] \right)^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} & (2p, (\mathbf{m}_{2p}, \xi_{2p-1}) | (\mathbf{m}_{2p}, \xi_{2p-1}); \epsilon) = \\ & = \prod_{r=1}^{p-1} [l_{r,2p-1} - \frac{1}{2}], \quad \text{if } m_{p,2p} = \frac{1}{2}. \end{aligned} \quad (40)$$

(They are defined up to normalization, that is, multiplication of these CGC's by some constants will not spoil the following results.)

All the other CGC's can be found from the just presented by the following formula:

$$\begin{aligned} & (k, \xi_n | \xi'_n; \epsilon) = q^{k-n} \times \\ & \times \frac{\langle \mathbf{m}_{n+1}, \xi_n | T_{\tilde{\epsilon}, \mathbf{m}_{n+1}}(I_{n+1, k}^-) | \mathbf{m}_{n+1}, \xi'_n \rangle}{\langle \mathbf{m}_{n+1}, \mathbf{m}_n, \xi_{n-1} | T_{\tilde{\epsilon}, \mathbf{m}_{n+1}}(I_{n+1, n}) | \mathbf{m}_{n+1}, \mathbf{m}'_n, \xi_{n-1} \rangle} \\ & \times (n, (\mathbf{m}_n, \xi_{n-1}) | (\mathbf{m}'_n, \xi_{n-1}); \epsilon), \end{aligned} \quad (41)$$

where the generators $I_{n+1, k}^-$ are defined in (2), $\tilde{\epsilon} = (\epsilon_2, \epsilon_3, \dots, \epsilon_n, +1)$. If $k = +$ or $k = -$ on the left-hand side of (41), one must put $k = 2$ on right-hand side. The set \mathbf{m}_{n+1} must be chosen to give the non-zero denominator on right-hand side of (41). Note that if $(n, (\mathbf{m}_n, \xi_{n-1}) | (\mathbf{m}'_n, \xi_{n-1})) \neq 0$, one can always do such a choice, moreover, the resulting CGC will not depend on this particular choice. In the case $n = 3$, we reobtain the CGC's corresponding to the nonclassical type representations for the algebra $U'_q(\mathfrak{so}_3)$ (see Section 7).

Theorem 2. *The formulas for the action of the operators $T^{\otimes}(I_{k+1, k})$, $k = 1, 2, \dots, n-1$, on the vectors $|\mathbf{m}'_n, \xi_{n-1}\rangle^{\otimes}$ defined by (38) with CGC's defined by (39)–(41), coincide with the corresponding formulas (20)–(21) for the action of the operators $T_{\epsilon, \mathbf{m}'_n}(I_{k+1, k})$ on the GT basis vectors $|\mathbf{m}'_n, \xi_{n-1}\rangle$. We have decomposition (34).*

9. The Wigner–Eckart Theorem for Vector Operators

To fix the idea, we restrict ourselves to the case where a vector operator acts on the space where direct sum of classical type representations of $U'_q(\mathfrak{so}_n)$ is realized.

Formula (28) give us the transformation from the basis $\{v_k \otimes |\xi_n\rangle\}$ to the basis $\{|\xi'_n\rangle^{\otimes}\}$ in the space \mathcal{V}^{\otimes} . Because of (27), the transformation matrix is a non-degenerate matrix with matrix elements being CGC's $(k, \xi_n | \xi'_n)$. Denote the matrix elements of the inverse matrix by $(\xi'_n | k, \xi_n)$ (inverse CGC's). Let us find the expression for the vector $v_n \otimes |\mathbf{m}_n, \xi_{n-1}\rangle$ from (28)

in terms of vectors $|\xi'_n\rangle^{\otimes}$. Since this vector transforms under the action of $T^{\otimes}(a)$, $a \in U'_q(\mathfrak{so}_{n-1})$, as the vector $|\xi_{n-1}\rangle$ under the action of $T_{\mathfrak{m}_{n-1}}(a)$ (see formula (5)), from the Schur lemma, we have

$$v_n \otimes |\mathfrak{m}_n, \xi_{n-1}\rangle = \sum_{\mathfrak{m}'_n} ((\mathfrak{m}'_n, \xi_{n-1})|n, (\mathfrak{m}_n, \xi_{n-1})) |\mathfrak{m}'_n, \xi_{n-1}\rangle^{\otimes}, \quad (42)$$

where the coefficients $((\mathfrak{m}'_n, \xi_{n-1})|n, (\mathfrak{m}_n, \xi_{n-1}))$ depend only on \mathfrak{m}'_n , \mathfrak{m}_n , \mathfrak{m}_{n-1} . From (28), it also follows that $\mathfrak{m}'_n \in \mathcal{S}(\mathfrak{m}_n)$. Although these coefficients are uniquely defined by (28)–(31), we shall need only their explicit dependence on \mathfrak{m}_{n-1} .

Definition 1. *The set $\{V_k\}$, $k = 1, 2, \dots, n$, of operators on \mathcal{V} , where a representation T of $U'_q(\mathfrak{so}_n)$ is realized, such that*

$$[V_{j-1}, T(I_{j,j-1})]_q = V_j, \quad [T(I_{j,j-1}), V_j]_q = V_{j-1}, \quad (43)$$

$$[T(I_{j,j-1}), V_k] = 0, \quad \text{if } j \neq k \text{ and } j - 1 \neq k, \quad (44)$$

where $[X, Y]_q = q^{1/2}XY - q^{-1/2}YX$, is called the vector operator of the algebra $U'_q(\mathfrak{so}_n)$.

It is easy to verify, that the action of the operators $T(I_{j,j-1})$ on the vectors $V_k v_\alpha$ directly corresponds to action (3)–(5) of operators $T^{\otimes}(I_{j,j-1})$ on the vectors $v_k \otimes v_\alpha$.

Let T be a direct sum of irreducible classical type representations of $U'_q(\mathfrak{so}_n)$ with arbitrary multiplicities. Choose the Gel'fand–Tsetlin (GT) basis in \mathcal{V} . Let us consider an invariant subspace $\mathcal{V}_{\mathfrak{m}_n, s}$ where a subrepresentation equivalent to $T_{\mathfrak{m}_n}$ is realized. The number s labels the number of such a subspace if the corresponding multiplicity exceeds 1. Combine the vectors $V_k |(\mathfrak{m}_n, \xi_{n-1}); s\rangle$, where $\{|(\mathfrak{m}_n, \xi_{n-1}); s\rangle\}$ is GT basis of $\mathcal{V}_{\mathfrak{m}_n, s}$, with CGC as in (28) for some fixed $\mathfrak{m}'_n \in \mathcal{S}(\mathfrak{m}_n)$. Two variants are possible. First, all the vectors $|\mathfrak{m}'_n, \xi_{n-1}\rangle^{\otimes}$ are zero. Second, on the space spanned by the vectors $|\mathfrak{m}'_n, \xi_{n-1}\rangle^{\otimes}$, a representation of $U'_q(\mathfrak{so}_n)$ equivalent to $T_{\mathfrak{m}'_n}$ is realized. From the Schur lemma, it follows that

$$|\mathfrak{m}'_n, \xi_{n-1}\rangle^{\otimes} = \sum_{s'} (\mathfrak{m}'_n, s' \|V\| \mathfrak{m}_n, s) |\mathfrak{m}'_n, \xi_{n-1}; s'\rangle, \quad (45)$$

where $(\mathfrak{m}'_n, s' \|V\| \mathfrak{m}_n, s)$ are some coefficients (*reduced matrix elements*) depending only on \mathfrak{m}'_n , s' , \mathfrak{m}_n , s and the vector operator $\{V_k\}$. Using the analogue of relation (42) for the vector operator and (45) we have

$$V_n |\mathfrak{m}_n, \xi_{n-1}; s\rangle = \sum_{\mathfrak{m}'_n, s'} ((\mathfrak{m}'_n, \xi_{n-1})|n, (\mathfrak{m}_n, \xi_{n-1})) \times$$

$$\times (\mathfrak{m}'_n, s' \|V\| \mathfrak{m}_n, s) |\mathfrak{m}'_n, \xi_{n-1}; s'\rangle. \quad (46)$$

As claimed above, the coefficients $((\mathfrak{m}'_n, \xi_{n-1})|n, (\mathfrak{m}_n, \xi_{n-1}))$ may depend on \mathfrak{m}_{n-1} . Since this dependence is identical for all the possible vector operators in arbitrary spaces, we choose, for a moment, \mathcal{V} to be the space $\mathcal{V}_{\mathfrak{m}_{n+1}}$ of the irreducible representation $T_{\mathfrak{m}_{n+1}}$ of $U'_q(\mathfrak{so}_{n+1})$ for some convenient \mathfrak{m}_{n+1} , and $\{V_k\} \equiv \{T_{\mathfrak{m}_{n+1}}(I_{n+1, k}^+)\}$. Extracting the dependence on \mathfrak{m}_{n-1} from the matrix elements of $T_{\mathfrak{m}_{n+1}}(I_{n+1, n})$ and comparing it with formulas (29)–(30), we obtain

$$((\mathfrak{m}'_n, \xi_{n-1})|n, (\mathfrak{m}_n, \xi_{n-1})) = (n, (\mathfrak{m}_n, \xi_{n-1}) | (\mathfrak{m}'_n, \xi_{n-1})) \lambda_{\mathfrak{m}'_n, \mathfrak{m}_n},$$

where $\lambda_{\mathfrak{m}'_n, \mathfrak{m}_n}$ are some coefficients depending on \mathfrak{m}'_n and \mathfrak{m}_n only. Returning to formula (46) and denoting $(\mathfrak{m}'_n, s' \|V\| \mathfrak{m}_n, s)' = (\mathfrak{m}'_n, s' \|V\| \mathfrak{m}_n, s) \times \lambda_{\mathfrak{m}'_n, \mathfrak{m}_n}$, we have

$$V_n |\mathfrak{m}_n, \xi_{n-1}; s\rangle = \sum_{\mathfrak{m}'_n, s'} (n, (\mathfrak{m}_n, \xi_{n-1}) | (\mathfrak{m}'_n, \xi_{n-1})) \times (\mathfrak{m}'_n, s' \|V\| \mathfrak{m}_n, s)' |\mathfrak{m}'_n, \xi_{n-1}; s'\rangle. \quad (47)$$

Iterating the second formula in (43), we obtain the action formulas for $\{V_k\}$, $1 \leq k < n$. Thus, we deduce the following q -analogue of the Wigner–Eckart theorem.

Theorem 4. *If \mathcal{V} is a Hilbert space and its Gel'fand–Tsetlin basis $\{|\mathfrak{m}_n, \xi_{n-1}; s\rangle\}$ is orthonormal, we have, for the components of a vector operator $\{V_k\}$ on \mathcal{V} , the decomposition*

$$\langle \mathfrak{m}'_n, \xi'_{n-1}; s' | V_k | \mathfrak{m}_n, \xi_{n-1}; s \rangle = ((\mathfrak{m}'_n, \xi'_{n-1}) | k, (\mathfrak{m}_n, \xi_{n-1}))' (\mathfrak{m}'_n, s' \|V\| \mathfrak{m}_n, s)',$$

where

$$((\mathfrak{m}'_n, \xi'_{n-1}) | k, (\mathfrak{m}_n, \xi_{n-1}))' = \frac{\langle \mathfrak{m}_{n+1}, \xi'_n | T_{\mathfrak{m}_{n+1}}(I_{n+1, k}^+) | \mathfrak{m}_{n+1}, \xi_n \rangle}{\langle \mathfrak{m}_{n+1}, \mathfrak{m}'_n, \xi_{n-1} | T_{\mathfrak{m}_{n+1}}(I_{n+1, n}) | \mathfrak{m}_{n+1}, \mathfrak{m}_n, \xi_{n-1} \rangle} \times (n, (\mathfrak{m}_n, \xi_{n-1}) | (\mathfrak{m}'_n, \xi_{n-1})), 1 \leq k < n$$

(see the comments after the analogous formula (31)).

Let T_ϵ be a direct sum of irreducible nonclassical type representations of $U'_q(\mathfrak{so}_n)$ with arbitrary multiplicities and fixed ϵ on the Hilbert space \mathcal{V}_ϵ . Choose the Gel'fand–Tsetlin basis in \mathcal{V}_ϵ . The space \mathcal{V}_ϵ is a direct sum of subspaces $\mathcal{V}_{\epsilon, \mathfrak{m}_n, s}$, where subrepresentations equivalent

to $T_{\epsilon, \mathbf{m}_n}$ are realized. The number s labels the number of such a subspace if the corresponding multiplicity exceeds 1. Using the argumentation analogous to the case of classical type representations, we derive the following q -analogue of the Wigner-Eckart theorem in the case of nonclassical type representations.

Theorem 5. *If \mathcal{V}_ϵ is a Hilbert space and its Gel'fand-Tsetlin basis $\{|\mathbf{m}_n, \xi_{n-1}; s\rangle\}$ is orthonormal, we have, for the components of a vector operator $\{V_k\}$ on \mathcal{V}_ϵ , the decomposition*

$$\begin{aligned} & \langle \mathbf{m}'_n, \xi'_{n-1}; s' | V_k | \mathbf{m}_n, \xi_{n-1}; s \rangle = \\ & = ((\mathbf{m}'_n, \xi'_{n-1}) | k, (\mathbf{m}_n, \xi_{n-1}); \epsilon)' \times \\ & \times (\epsilon, \mathbf{m}'_n, s' | | V | | \epsilon, \mathbf{m}_n, s)', \end{aligned}$$

where

$$\begin{aligned} & ((\mathbf{m}'_n, \xi'_{n-1}) | k, (\mathbf{m}_n, \xi_{n-1}); \epsilon)' = \\ & = \frac{\langle \mathbf{m}_{n+1}, \xi'_{n-1} | T_{\bar{\epsilon}, \mathbf{m}_{n+1}}(I_{n+1, k}^+) | \mathbf{m}_{n+1}, \xi_n \rangle}{\langle \mathbf{m}_{n+1}, \mathbf{m}'_n, \xi_{n-1} | T_{\bar{\epsilon}, \mathbf{m}_{n+1}}(I_{n+1, n}) | \mathbf{m}_{n+1}, \mathbf{m}_n, \xi_{n-1} \rangle} \\ & \times (n, (\mathbf{m}_n, \xi_{n-1}) | (\mathbf{m}'_n, \xi_{n-1}); \epsilon), 1 \leq k < n \end{aligned}$$

(see the comments after the analogous formula (41)).

The coefficients $(\epsilon, \mathbf{m}'_n, s' | | V | | \epsilon, \mathbf{m}_n, s)'$ are *reduced matrix elements* for the vector operator $\{V_k\}$.

If the representation T is a direct sum of classical type representations and nonclassical type representations with different ϵ , it is easy to find the matrix elements for vector operators. It is sufficient to take into account the fact that a vector operator 'acting' on a classical type representation can not give a nonclassical type representation, and 'acting' on a nonclassical type representation with some set ϵ can not give a nonclassical type representation with other set ϵ' . Thus, the corresponding matrix elements are zero. The non-zero matrix elements are described by Theorem 4 and Theorem 5.

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ТЕОРЕМА ВІГНЕРА-ЕККАРТА ДЛЯ ДЕЯКОЇ АЛГЕБРИ, ЩО З'ЯВЛЯЄТЬСЯ В КВАНТОВІЙ ГРАВИТАЦІЇ

М.З.Іоргов

Р е з ю м е

Визначено тензорний добуток векторного та довільного представлень нестандартної q -деформації $U'_q(\mathfrak{so}_n)$ універсальної обвідної алгебри $U(\mathfrak{so}_n)$ алгебри Лі \mathfrak{so}_n . Відомо, що ця алгебра з'являється в $(2+1)$ -вимірній квантовій гравітації. Знайдено в явном вигляді коефіцієнти Клебша-Гордана тензорного добутку векторного та довільного (класичного або некласичного типу) представлень q -алгебри $U'_q(\mathfrak{so}_n)$. Доведено теорему Вігнера-Еккарта для векторних операторів.

ТЕОРЕМА ВІГНЕРА-ЭККАРТА ДЛЯ НЕКОТОРОЙ АЛГЕБРЫ, ПОЯВЛЯЮЩЕЙСЯ В КВАНТОВОЙ ГРАВИТАЦИИ

Н.З.Иоргов

Р е з ю м е

Определено тензорное произведение векторного и произвольного представлений нестандартной q -деформации $U'_q(\mathfrak{so}_n)$ универсальной обертывающей алгебры $U(\mathfrak{so}_n)$ алгебры Ли \mathfrak{so}_n . Известно, что эта алгебра появляется в $(2+1)$ -мерной квантовой гравитации. Найден в явном виде коэффициенты Клебша-Гордана тензорного произведения векторного и произвольного (классического или неклассического типа) представлений q -алгебры $U'_q(\mathfrak{so}_n)$. Доказана теорема Вигнера-Эккарта для векторных операторов.