QUANTIZATION OF NONCOMMUTATIVE FIELD THEORIES[1]

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In this talk, the quantization of noncommutative field theories is reviewed. Particular weight is given to the convergence theorem by Chepelev and Roiban. Several examples of renormalizable noncommutative field theories are given.

1. Introduction

The idea of using non-commuting coordinates as a method to quantize space and time goes back to Snyder [1]. More recently, this idea has become popular due to the appearance of noncommutative space-time in a 'stringy setting' [2]. However, non-commuting coordinates are interesting in their own right, especially seen in the light of the work by Connes and co-authers [3], who managed to relate the standard model and gravity to a partly noncommutative geometry.

In this short talk, I will consider field theory constructed on a noncommutative algebra involving a constant noncommutative parameter. Since this parameter will have dimension -2, the corresponding field theory will be power counting non-renormalizable. Thus, the question of renormalization is *a priori* of fundamental importance. I will give a short review of the present state of the art as well as a few examples of renormalizable noncommutative field theories.

2. Noncommutative Field Theory

It is assumed that the smooth structure of space-time will break down at some scale (Planck scale) , rendering a noncommutative algebra generated by 'coordinates' $\hat{x}^\mu$, and the commutator relation

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} \ . \tag{1}$$

This algebraic structure may be represented via a $\star$-product, which may be given explicitely in this case of a constant θ as

$$(f \star g)(x) = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \times$$

$$\times \mathrm{e}^{-\mathrm{i}(k_\mu + p_\mu)x^\mu} \, \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu}k_\mu p_\nu} \tilde{f}(k) \, \tilde{g}(p) \ , \tag{2}$$

where $f(x), g(x)$ are functions on a commutative manifold. Note that the $\star$-product is nonlinear. The $\star$-product (2) reassembles the algebraic relation (1) on a commutative manifold

$$[x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu} \ . \tag{3}$$

One may now define a noncommutative field theory by constructing an action in the usual manner and replacing usual products with the $\star$-product. However, in the case of gauge theories, special care must be made [4]. The $\star$-product is cyclic invariant:

$$\int d^Dx \, (\phi_1 \star \phi_2 \star \ldots \phi_n)(x) =$$

$$= \int d^Dx \, (\phi_2 \star \ldots \phi_n \star \phi_1)(x) \ , \tag{4}$$

which very much resembles the cyclic property of the trace. This observation lies at the heart of the non-Abelian structure appearing in noncommutative Abelian gauge theories [4].

Also, using (6), one may perform the functional derivation

$$\frac{\delta}{\delta\phi_1(y)} \int d^Dx \, (\phi_1 \star \phi_2 \star \ldots \phi_n)(x) =$$

$$= (\phi_2 \star \ldots \phi_n)(y) \, \delta(x-y). \tag{5}$$

Properties (6) and (5) suggest that commutative (classical) symmetries such as gauge- (BRST-) and super-symmetries will persist in some sense also in the noncommutative realm.

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3. Feynman Graphs and Loop Calculations

3.1. Feynman Rules

In order to check the perturbative UV and IR behaviour of a noncommutative field theory, one needs the corresponding Feynman rules. Due to the identity

$$\int d^D x \, (\phi_1 \star \phi_2)(x) = \int d^D x \, (\phi_1 \phi_2)(x) \ , \tag{6}$$

which states that the θ-pendency of the $\star$-product vanishes for bilinear terms, one finds that the propagators are unchanged. For the noncommutative vertices V_{nc} , $\star$-product (2) yields a phase

$$V_{nc}(k_1, \dots, k_n) = \frac{1}{n}\delta(k_1 + \dots + k_n)\times$$

$$\times \exp(\mathrm{i}\sum_{i<j} \theta^{\alpha\beta} k_{i,\alpha} k_{j,\beta}) \ . \tag{7}$$

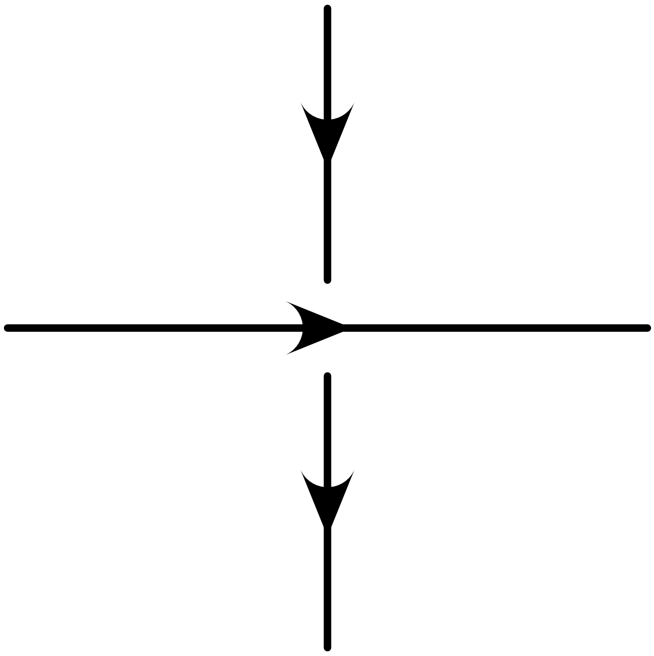
It is the presence of phase (7) which makes noncommutative field theories of fundamental interest. Indeed, one could hope that the total phase obtained in a Feynman graph would give a damping effect which could render the graph UV finite. In the following, we shall see that this is only partly true.

3.2. Feynman Graphs

With the above set of Feynman rules one finds that the noncommutative effect on the integrand of a Feynman graph I_{nc} sums up to a phase

$$I_{nc} = \exp(i\varphi) I \ , \tag{8}$$

where I is the integrand valid in the commutative case and φ is some phase. However, it was shown by Filk [5] that for planar, diagrams, i.e. diagrams without crossing of lines, the phase only depends on the external momenta. Only in nonplanar diagrams, i.e. diagrams with crossing momenta lines (see Figure), one obtains a phase involving internal loop momenta. This difference is crucial: Planar graphs will display exactly the same convergency behaviour as in the commutative case, whereas nonplanar graphs will (to some extent) be damped by the phase φ. It is therefore important to distinguish between the planar and nonplanar sectors of perturbative analysis of a noncommutative field theory. Furthermore, since nonplanar graphs are damped with



Line-crossing in a non-planar graph

a factor proportional to $\tilde{p}^\alpha = \theta^{\alpha\beta} p_\beta$, where p_β is an external momenta, we see that the damping effect is proportional to $\tilde{p}^\alpha$. This means that the UV divergency reappears as an IR divergency in the limit $p \to 0$. This effect is known as UV/IR mixing. The degree of IR divergency is inverse proportional to the degree of the original UV divergency.

4. Convergence Theorem

A general convergence theorem for noncommutative field theory was given by Chepelev and Roiban [6]. Furthermore, they have given a noncommutative version of the BPHZ subtraction scheme (valid for divergent planar graphs).

The convergency theorem roughly states that a 1PI graph G is convergent if, for any subgraph γ of G, the following relation holds

$$\omega(\gamma) - c_G(\gamma) d < 0 \ , \tag{9}$$

where $c_G(\gamma)$ is the number of nontrivial homology cycles of $\Sigma(G)$, whereby $\Sigma(G)$ is the genus g 2-surface (with boundary), on which G may be written. $\omega(G)$ is the (commutative) degree of divergency,

$$\omega(G) = dL(G) - 2I(G) \ , \tag{10}$$

where d is the space-time dimension, L – number of loops and I – the number of internal lines of the graph G.

Now, it is important to note that the noncommutative BPHZ subtraction scheme given in [6] only works for planar graphs. It is therefore crucial for the renormalizability of a noncommutative field theory that divergent non-planar diagrams do not occur. This is the case if:

- All graphs are at most logarithmic divergent

$$\omega(\gamma) \leq 0 \quad \forall\gamma \, . \tag{11}$$

- Strong symmetries are present in the theory cancelling 'bad' divergencies.

Till now we have only considered the UV regime. However, one must also take into account the IR behaviour of (nested) nonplanar graphs. I shall not deal with this problem in this talk. In the following sections, a few examples of renormalizable noncommutative field theories are given.

5. The Noncommutative Chern–Simons Theory in $D = 3$

A nice example of a renormalizable noncommutative field theory is the noncommutative Chern–Simons model in three dimensions. It is given by the action

$$S_{cl} = -\frac{1}{2}\int d^3x\ \epsilon_{\mu\nu\rho}\mathrm{tr}\Big(A_\mu\partial_\nu A_\rho -$$

$$-\frac{2\mathrm{i}}{3}A_\mu \star A_\nu \star A_\rho\Big), \tag{12}$$

where $\epsilon_{\mu\nu\rho}$ is the 3-dimensional totally antisymmetric symbol. We have:

- The model is topological: Independent of the metric.
- The equation of motion has the simple form $F_{\mu\nu} = 0$.
- The model has a vanishing energy-momentum tensor, which has the form of a total BRST variation upon gauge fixing. This again leads to the existence of an 'odd' version of the translational generator P_μ, the vector-supersymmetry:

$$T_{\mu\nu} = 0 \quad \Rightarrow \quad T_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\} \, , \tag{13}$$

$$\Rightarrow \quad P_\mu = \int d^2x\ T_{0\mu} = \{Q, V_\mu\} \, . \tag{14}$$

The vector-supersymmetry is responsible for the renormalizability of the theory since it leads to the full cancelation of all divergent graphs. This effect is also found in the noncommutative case. A full proof of this can be found in [7].

6. Noncommutative Wess–Zumino Model in $D = 4$

The Wess–Zumino model is most conveniently treated in the superfield formalism, where it is characterized by the classical action

$$S_{\rm cl} = \frac{1}{16}\int dV\ \bar{\phi}\phi + \frac{m}{8}\left[\int dS\ \phi\phi + d\bar{S}\ \bar{\phi}\bar{\phi}\right] +$$

$$+\frac{g}{48}\left[\int dS\ \phi\star\phi\star\phi + \int d\bar{S}\ \bar{\phi}\star\bar{\phi}\star\bar{\phi}\right] \, , \tag{15}$$

with $dV = d^4x\ D^2\bar{D}^2$, $dS = d^4x\ D^2$, $\bar{S} = d^4x\ \bar{D}^2$. The fields ϕ and $\bar{\phi}$ are chiral and anti-chiral superfields

$$\phi \;=\; A + \theta^\alpha\psi_\alpha + \theta^\alpha\theta_\alpha F \tag{16}$$

$$\bar{\phi} \;=\; \bar{A} + \bar{\theta}^{\dot\alpha}\bar{\psi}_{\dot\alpha} + \bar{\theta}^{\dot\alpha}\bar{\theta}_{\dot\alpha}F \, . \tag{17}$$

The Wess–Zumino model displays only logarithmic divergencies. This means that no divergent non-planar diagrams will appear in the noncommutative Wess–Zumino model, and one may conclude that the model is renormalizable [8].

7. Noncommutative Supersymmetric Yang–Mills Theory

Noncommutative Yang–Mills theory displays divergencies of higher order than logarithmic. Also, problematic IR problems occur [6, 9]. A possible solution is to consider a supersymmetric theory: Noncommutative Supersymmetric Yang–Mills. Again it is preferable to consider a superfield formalism:

$$\phi = \phi^a T^a \, ,$$

$$\phi^a = c^a + \theta^\alpha\xi^a_\alpha + \ldots \qquad (\text{real}) \, ,$$

$$S_{cl} = -\frac{1}{128}\mathrm{tr}\int dS\ F^\alpha \star F_\alpha \, ,$$

$$F_\alpha = \bar{D}\bar{D}\left(e^\phi D_\alpha e^\phi\right)^\star \, , \tag{18}$$

where T^a is a generator of the gauge group and ϕ a real superfield. Work is still in progress. Due to the presence of a supersymmetry, we expect to find only logarithmic divergencies and thus a renormalizable model.

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КВАНТУВАННЯ НЕКОМУТАТИВНИХ ТЕОРІЙ ПОЛЯ

Й. Грімштруп

Р е з ю м е

Зроблено огляд робіт з квантування некомутативних теорій поля. Зокрема, особливу увагу приділено теорії збіжності Чепелева й Ройбана. Наведено багато прикладів перенормування некомутативних теорій поля.

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Сделан обзор работ по квантованию некоммутативных теорий поля. В частности, особое внимание уделено теореме сходимости Чепелева и Ройбана. Приведено много примеров перенормирования некоммутативных теорий поля.