THE ENERGY-MOMENTUM TENSOR ON NONCOMMUTATIVE SPACES—SOME PEDAGOGICAL COMMENTS[1]

A. GERHOLD, J. GRIMSTRUP, H. GROSSE[1], L. POPP, M. SCHWEDA, R. WULKENHAAR[1]

Institut für Theoretische Physik, Technische Universität Wien
(Wiedner Hauptstraße 8–10, A-1040 Wien, Austria),

[1] Institut für Theoretische Physik, Universität Wien
(Boltzmanngasse 5, A-1090 Wien, Austria)

UDC 538.9; 538.915; 517.957

We present the discussion of the energy-momentum tensor of the scalar ϕ^4-theory on a noncommutative space. The Noether procedure is performed at the operator level. Additionally, the broken dilatation symmetry will be considered in a Moyal—Weyl deformed scalar field theory at the classical level.

1. Introduction

The aim of this work is to investigate the translation and dilatation symmetry at least at the classical level for a noncommutative ϕ^4-theory. Not much work has yet been done in this direction, only in [1] and [2] one finds some scattered remarks concerning the energy-momentum tensor and its Noether procedure for Moyal—Weyl deformed scalar field theories. In this paper, we extend the analysis of [2] and formulate the Noether procedure for translations already at the operator level. By the use of the Moyal—Weyl correspondence between operators and fields, we are able to confirm the results of [2].

This work is organized as follows. Section 2 is devoted to some special features of the quantum space in connection with a ϕ^4-theory.

In Section 3, we study the construction of the energy-momentum tensor at the operator level and in a Moyal—Weyl deformed ϕ^4-theory.

Finally, in the last section, we investigate the broken dilatation symmetry of the noncommutative ϕ^4-theory.

2. The Quantum Phase Space and the Scalar Field Theory

We consider the scalar field theory which is described at the classical level by the following action[2]:

$$S^{(0)}[\phi] = \int dx \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right), \qquad (1)$$

where $\phi(x)$ is a real valued scalar field on the four-dimensional Euclidean space E_4. For fields in a Schwartz space of functions which decrease sufficiently fast at infinity, we may define a Fourier transformation by

$$\phi(x) = \int dk\, e^{ik_\mu x_\mu}\tilde{\phi}(k),$$
$$\tilde{\phi}(k) = \int dx\, e^{-ik_\mu x_\mu}\phi(x), \qquad (2)$$

with $\tilde{\phi}(-k) = \tilde{\phi}^*(k)$. In order to generalize a field theory on an ordinary space to one on a noncommutative space, we replace the local coordinates x_μ by Hermitian operators $\hat{x}_\mu$ obeying the relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\sigma_{\mu\nu}, \quad [\hat{x}_\mu, \sigma_{\mu\nu}] = 0, \qquad (3)$$

where $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$ is a real invertible matrix. Consequently, fields on space-time are replaced by operators. Replacing x_μ by $\hat{x}_\mu$ in (2), we obtain

$$\phi(\hat{x}_\mu) = \int dk\, e^{ik\hat{x}}\tilde{\phi}(k). \qquad (4)$$

With (2), one gets

$$\phi(\hat{x}) = \int dk \int dx\, \phi(x) e^{ik\hat{x}-ikx} =$$

[1]Presented at the XIIIth International Hutsulian Workshop "Methods of Theoretical and Mathematical Physics" (September 11 — 24 2000, Uzhgorod — Kyiv — Ivano-Frankivsk — Rakhiv, Ukraine) and dedicated to Prof. Dr. W. Kummer on the occasion of his 65th birthday.

[2]We use the shorthand notations $dx := d^4x$ and $dk := \frac{d^4k}{(2\pi)^4}$.

$$= \int dx \int dk\, \hat{T}(k) e^{-ikx} \phi(x) = \int dx \hat{\Delta}(x) \phi(x). \tag{5}$$

$\phi(\hat{x})$ is an element of an algebra $\mathcal{A}_x$ in the sense of [6].

In (5), we have introduced the operators $\hat{T}(k)$ and $\hat{\Delta}(k)$ which were originally defined by Balasz et al. [3]. More recently, these operators were also used by Filk [4]:

$$\hat{T}(k) = e^{ik\hat{x}}, \tag{6}$$

and by Ambjorn et al. [5]:

$$\hat{\Delta}(x) = \int dk\, e^{ik\hat{x} - ikx} = \int dk\, \hat{T}(k) e^{-ikx}. \tag{7}$$

$\hat{T}(k)$ and $\hat{\Delta}(x)$ have different useful properties for practical calculations. In order to list these properties, let us define the trace operations for $\hat{T}(k)$ and $\hat{\Delta}(x)$.

For simplicity, we choose the space dimension $d = 2$ and consider the trace of $\hat{T}(k)$ in a first step. The operator $\hat{T}(k)$ has the following properties [4]:

$$\hat{T}^\dagger(k) = \hat{T}(-k),$$

$$\hat{T}(k)\hat{T}(k') = e^{-ik\times k'} \hat{T}(k + k'), \tag{8}$$

where $k \times k' := \frac{1}{2}\sigma_{\mu\nu} k_\mu k'_\nu$. For $d = 2$, we have

$$\sigma_{\mu\nu} = \sigma\varepsilon_{\mu\nu} = \sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{9}$$

and Eq. (3) becomes

$$[\hat{x}_1, \hat{x}_2] = i\sigma. \tag{10}$$

The following remarks concerning the definitions of traces can be deduced with the methods of [3]. Eq. (10) looks like the usual commutation relation of ordinary quantum mechanics between $\hat{q}$ and $\hat{p}$ if one identifies $\hat{x}_1 = \hat{q}$ and $\hat{x}_2 = \hat{p}$. The corresponding eigenstates are defined by [3]:

$$\hat{x}_1|x\rangle = x|x\rangle,$$
$$\hat{x}_2|p\rangle = \sigma p|p\rangle \tag{11}$$

with

$$\langle x|x'\rangle = \delta(x - x'), \quad \int dx\, |x\rangle\langle x| = 1,$$
$$\langle p|p'\rangle = \delta(p - p'), \quad \int dp\, |p\rangle\langle p| = 1, \tag{12}$$

and

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipx}. \tag{13}$$

Now it is straightforward to calculate the matrix elements of $\hat{T}(k)$. The result is

$$\langle x'|\hat{T}(k)|x''\rangle = \delta(k_2\sigma + x' - x'') e^{ik_1(x'+x'')/2}, \tag{14}$$

implying that the trace is (with an appropriate normalization factor)

$$\mathrm{Tr}\, \hat{T}(k) := 2\pi\sigma \int dx\, \langle x|T(k)|x\rangle =$$

$$= 2\pi\sigma\, \delta(k_2\sigma) \int dx\, e^{ik_1 x} = (2\pi)^2 \delta^{(2)}(k_\mu). \tag{15}$$

With Eq. (14), we are also able to calculate the matrix elements of $\hat{\Delta}(x)$. A short calculation gives

$$\langle x'|\hat{\Delta}(x)|x''\rangle = \int dk\, \langle x'|\hat{T}(k)|x''\rangle e^{-ikx} =$$

$$= \frac{1}{2\pi\sigma}\, \delta\left(x_1 - \frac{x' + x''}{2}\right) e^{i(x'-x'')x_2/\sigma}, \tag{16}$$

and the trace of $\hat{\Delta}(x)$ becomes

$$\mathrm{Tr}\, \hat{\Delta}(x) := 2\pi\sigma \int dx\, \langle x|\hat{\Delta}(x)|x\rangle =$$

$$= \int dx\, \delta(x_1 - x) = 1. \tag{17}$$

Eqs. (15) and (17) confirm the results of [4, 5]. Additionally, one can derive the relations

$$\mathrm{Tr}\, [\hat{T}(k)\hat{T}(k')] = (2\pi)^2 e^{-ik\times k'} \delta^{(2)}(k_\mu + k'_\mu) =$$

$$= (2\pi)^2\, \delta^{(2)}(k_\mu + k'_\mu),$$

$$\mathrm{Tr}\, [\hat{\Delta}(x)\hat{\Delta}(x')] = \delta^{(2)}(x_\mu - x'_\mu). \tag{18}$$

In order to be complete, we present an alternative way of calculating the trace of $\hat{T}(k)$:

$$\mathrm{Tr}\, \hat{T}(k) = 2\pi\sigma \int dx'\, \langle x'|\hat{T}(k)|x'\rangle, \tag{19}$$

where $|x'\rangle$ is now an appropriate representation of algebra (3):

$$[\hat{x}_\mu, \hat{x}_\nu]|x'\rangle = i\sigma\varepsilon_{\mu\nu}|x'\rangle. \tag{20}$$

A possible solution for (20) is

$$\hat{x}_\mu|x'\rangle = \left(x_\mu + \frac{i}{2}\sigma_{\mu\rho}\partial_\rho\right)|x'\rangle. \tag{21}$$

However, in this case, x' cannot be identified with x_μ due to the fact that x_μ and ∂_ρ represent $2 \times d$ degrees of freedom ($d = 2$). Therefore, we need an irreducible representation which eliminates the redundant degrees of freedom.

An irreducible representation is given by (renaming $x' \to x$):

$$\hat{x}_1|x\rangle = x|x\rangle,$$

$$\hat{x}_2|x\rangle = -i\sigma\frac{d}{dx}|x\rangle. \tag{22}$$

Using the Baker-Campbell-Hausdorff-formula and the fact that

$$e^{\sigma k_2 \frac{d}{dx}}|x\rangle = |x - \sigma k_2\rangle, \tag{23}$$

one obtains again (15).

In order to define a scalar field theory at the operator level, we need a derivation prescription [5—7]:

$$\hat{\partial}_\mu\phi(\hat{x}) = -i[\hat{x}'_\mu, \phi(\hat{x})] = \int dx\, \partial_\mu\phi(x)\hat{\Delta}(x), \tag{24}$$

where $\hat{x}'_\mu = \sigma^{-1}_{\mu\nu}\hat{x}_\nu$ and $\sigma_{\mu\rho}\sigma^{-1}_{\rho\nu} = \delta_{\mu\nu}$. This definition implies

$$[\hat{\partial}_\mu, \hat{x}_\nu] = \delta_{\mu\nu}, \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0. \tag{25}$$

Furthermore, we have the Leibniz rule

$$\hat{\partial}_\mu(f(\hat{x})g(\hat{x})) = \hat{\partial}_\mu f(\hat{x})g(\hat{x}) + f(\hat{x})\hat{\partial}_\mu g(\hat{x}). \tag{26}$$

Additionally, one can show that one has the following useful relation:

$$\left[\hat{\partial}_\mu, \hat{\Delta}(x)\right] = -\partial_\mu\hat{\Delta}(x). \tag{27}$$

Eq. (27) implies

$$e^{-v_\mu\hat{\partial}_\mu}\hat{\Delta}(x)e^{v_\mu\hat{\partial}_\mu} = \hat{\Delta}(x+v). \tag{28}$$

The existence of such an operator implies that $\mathrm{Tr}\,\hat{\Delta}(x)$ is independent of x for any trace operation on the algebra of operators. (28) gives therefore

$$\mathrm{Tr}\,\hat{\Delta}(x) = \mathrm{Tr}\hat{\Delta}(x+v) \tag{29}$$

and thus one has in consistency with (17):

$$\mathrm{Tr}\,\phi(\hat{x}) = \int dx\, \phi(x)\mathrm{Tr}\,\hat{\Delta}(x) =$$

$$= \mathrm{Tr}\,\hat{\Delta}(x)\int dx\, \phi(x). \tag{30}$$

In normalizing $\mathrm{Tr}\,\hat{\Delta}(x)$ to one, we get

$$\mathrm{Tr}\,\phi(\hat{x}) = \int dx\, \phi(x). \tag{31}$$

Now we are able to define the inverse map of (5). In Filk's [4] notation, one obtains

$$\phi(x) = \int dk\, e^{ikx}\,\mathrm{Tr}\,[\phi(\hat{x})T^\dagger(k)] \tag{32}$$

and, corresponding to Ambjorn et al. [5], one has

$$\phi(x) = \mathrm{Tr}[\phi(\hat{x})\hat{\Delta}(x)], \tag{33}$$

allowing now to define a Moyal—Weyl product [4, 5] in the following manner:

$$(\phi_1 * \phi_2)(x) := \int dk\, e^{ikx}\,\mathrm{Tr}\,[\phi_1(\hat{x})\phi_2(\hat{x})T^\dagger(k)] =$$

$$= \mathrm{Tr}\,[\phi_1(\hat{x})\phi_2(\hat{x})\hat{\Delta}(x)] =$$

$$= \int dk_1 \int dk_2\, e^{i(k_1+k_2)x}e^{-ik_1\times k_2}\tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2). \tag{34}$$

Eqs. (33) and (34) show that there is a one-to-one correspondence between fields (of sufficiently rapid decrease at infinity) and operators. From (34), it follows also

$$\int dx\, (\phi_1 * \phi_2)(x) = \int dx\, \phi_1(x)\phi_2(x). \tag{35}$$

Furthermore one has

$$\mathrm{Tr}\,[\phi_1(\hat{x})\phi_2(\hat{x})] = \int dx\, (\phi_1 * \phi_2)(x). \tag{36}$$

and

$$\mathrm{Tr}[\phi(\hat{x})^4] = \int dx\, (\phi(x))^4_*. \tag{37}$$

One can easily show that cyclic rotation is allowed:

$$\int dx\, (\phi_1 * \phi_2 * \ldots * \phi_n)(x) =$$

$$= \int dx\, (\phi_n * \phi_1 * \ldots * \phi_{n-1})(x). \tag{38}$$

Using now all these definitions, one is able to define a scalar field theory on a noncommutative space-time at the "algebra" level as

$$\bar{S}^{(0)}[\phi] = \mathrm{Tr}\left(\frac{1}{2}(\hat{\partial}_\mu \phi(\hat{x}))^2 + \frac{m^2}{2}\phi(\hat{x})^2 + \frac{\lambda}{4!}\phi(\hat{x})^4\right) =$$

$$= \mathrm{Tr}\left(\bar{\mathcal{L}}^{(0)}(\phi(\hat{x}))\right). \tag{39}$$

With the help of (24), the latter expression may be rewritten as a "Moyal—Weyl deformed" action:

$$S^{(0)}[\phi] = \int dx\left(\frac{1}{2}\partial_\mu\phi * \partial_\mu\phi + \frac{m^2}{2}\phi * \phi + \frac{\lambda}{4!}(\phi)^4_*\right) =$$

$$= \int dx\left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}(\phi)^4_*\right) =$$

$$= \int dx\, \mathcal{L}^{(0)}_*(\phi(x)). \tag{40}$$

We conclude this section with some remarks concerning the equation of motion at the algebra level. In order to see how this works, it is sufficient to discuss the free kinetic part

$$\bar{S}^{(0)}_{\text{free}}[\phi] = -\frac{1}{2}\mathrm{Tr}\left([\hat{x}'_\mu, \phi(\hat{x})][\hat{x}'_\mu, \phi(\hat{x})]\right) =$$

$$= \frac{1}{2}\mathrm{Tr}\,(\hat{\partial}_\mu\phi(\hat{x}))^2. \tag{41}$$

The "classical" equation of motion, similar to the commutative case, is obtained by minimizing the action:

$$\frac{\delta\bar{S}^{(0)}_{\text{free}}[\phi]}{\delta\phi(\hat{x})} = 0. \tag{42}$$

We define the functional derivative as usual [2]:

$$\bar{S}^{(0)}_{\text{free}}[\phi + \delta\phi] - \bar{S}^{(0)}_{\text{free}}[\phi] =: \mathrm{Tr}\left(\frac{\delta\bar{S}^{(0)}_{\text{free}}[\phi]}{\delta\phi(\hat{x})}\delta\phi(\hat{x})\right). \tag{43}$$

Using cyclic rotation, we obtain

$$\frac{\delta\bar{S}^{(0)}_{\text{free}}[\phi]}{\delta\phi(\hat{x})} = [\hat{x}'_\rho, [\hat{x}'_\rho, \phi(\hat{x})]] = -\hat{\partial}_\rho\hat{\partial}_\rho\phi(\hat{x}) = 0. \tag{44}$$

This is the massless free field equation of the theory. The inclusion of the mass term and the interaction gives the following equation of motion:

$$\frac{\delta\bar{S}^{(0)}[\phi]}{\delta\phi(\hat{x})} = -\hat{\partial}_\rho\hat{\partial}_\rho\phi(\hat{x}) + m^2\phi(\hat{x}) + \frac{\lambda}{3!}\phi(\hat{x})^3 = 0. \tag{45}$$

Eq. (45) will be used for the construction of the energy-momentum tensor in the next section. For the Moyal—Weyl deformed field theory one gets the equation of motion in a similar way [2]:

$$\frac{\delta S^{(0)}[\phi(x)]}{\delta\phi(x)} = -\partial_\rho\partial_\rho\phi(x) + m^2\phi(x) +$$

$$+ \frac{\lambda}{3!}(\phi)^3_*(x) = 0. \tag{46}$$

3. Noether Theorem for Translation Symmetry at the Algebra Level and its Moyal-deformed Counterpart

In order to define infinitesimal translations at the operator level, one generalizes the usual transformation law for a scalar field

$$\delta_\mu\phi(x) = \partial_\mu\phi(x) \tag{47}$$

to

$$\delta_\mu\phi(\hat{x}) = \hat{\partial}_\mu\phi(\hat{x}) = -i[\hat{x}'_\mu, \phi(\hat{x})] \tag{48}$$

in accordance with (24). Since the action

$$\bar{S}^{(0)}[\phi] = \mathrm{Tr}\left(\frac{1}{2}(\hat{\partial}_\mu\phi(\hat{x}))^2 + \frac{m^2}{2}\phi(\hat{x})^2 + \frac{\lambda}{4!}\phi(\hat{x})^4\right) \tag{49}$$

is invariant under translations, we can try to derive a Noether current in the following way. One calculates the variation of $\bar{S}^{(0)}[\phi(\hat{x})]$ in two different ways, once using the equation of motion and alternatively without using the equation of motion [9].

First we note that, with the help of (24) and performing cyclic rotations under the trace, one obtains the following formula for "partial integration":

$$\mathrm{Tr}\left(\phi_1(\hat{x})\hat{\partial}_\mu\phi_2(\hat{x})\right) = -\mathrm{Tr}\left(\hat{\partial}_\mu\phi_1(\hat{x})\phi_2(\hat{x})\right). \tag{50}$$

Then we have

$$\delta_\mu \bar{S}^{(0)}[\phi]_1 = \mathrm{Tr}\Big(\hat{\partial}_\mu\hat{\partial}_\rho\phi(\hat{x})\hat{\partial}_\rho\phi(\hat{x})+$$

$$+m^2\hat{\partial}_\mu\phi(\hat{x})\phi(\hat{x}) + \frac{\lambda}{3!}\hat{\partial}_\mu\phi(\hat{x})\phi^3(\hat{x})\Big) =$$

$$= \mathrm{Tr}\left(\hat{\partial}_\mu\mathcal{L}(\phi(\hat{x}))\right), \tag{51}$$

$$\delta_\mu \bar{S}^{(0)}[\phi]_2 = \mathrm{Tr}\Big(\hat{\partial}_\rho(\hat{\partial}_\mu\phi(\hat{x})\hat{\partial}_\rho\phi(\hat{x}))+$$

$$+\hat{\partial}_\mu\phi(\hat{x})\Big(-\hat{\partial}_\rho\hat{\partial}_\rho\phi(\hat{x}) + m^2\phi(\hat{x}) + \frac{\lambda}{3!}\phi(\hat{x})^3\Big)\Big). \tag{52}$$

Clearly, one has for the difference

$$\delta_\mu \bar{S}^{(0)}[\phi]_2 - \delta_\mu \bar{S}^{(0)}[\phi]_1 = 0. \tag{53}$$

This leads to

$$\mathrm{Tr}\left(\hat{\partial}_\rho\left[\frac{1}{2}(\hat{\partial}_\rho\phi(\hat{x})\hat{\partial}_\mu\phi(\hat{x}) + \hat{\partial}_\mu\phi(\hat{x})\hat{\partial}_\rho\phi(\hat{x}))-\right.\right.$$

$$\left.\left.-\delta_{\rho\mu}\bar{\mathcal{L}}(\phi(\hat{x}))\right]\right) = \mathrm{Tr}\left(\hat{\partial}_\rho T_{\rho\mu}\right) = 0, \tag{54}$$

where we have defined the (symmetrized) energy-momentum tensor at the algebra level

$$T_{\rho\mu}(\phi(\hat{x})) := \frac{1}{2}(\hat{\partial}_\rho\phi(\hat{x})\hat{\partial}_\mu\phi(\hat{x})+$$

$$+\hat{\partial}_\mu\phi(\hat{x})\hat{\partial}_\rho\phi(\hat{x})) - \delta_{\rho\mu}\bar{\mathcal{L}}^{(0)}. \tag{55}$$

It is important to note that Eq. (54) does not imply $\hat{\partial}_\rho T_{\rho\mu} = 0$ locally.

For the further discussion, we would like to perform a Wick rotation, i.e. the switch to the Minkowski space-time M_4. In order to do this and for reasons to be discussed below, we assume commutativity in the 4-component (which is to "become" time) $\sigma^{4i} = 0$, where i runs from 1 to 3.

Using the Moyal—Weyl prescription, one can rewrite (55) as[3]

$$T_{\rho\mu}(\phi(x)) = \frac{1}{2}(\partial_\rho\phi * \partial_\mu\phi + \partial_\mu\phi * \partial_\rho\phi) - \eta_{\rho\mu}\mathcal{L}_*^{(0)}. \tag{56}$$

Construction (56) is symmetric, therefore no Belinfante-procedure is needed [8]. Result (56) is consistent with

$$W_\mu S^{(0)}[\phi] = \int dx\, \partial_\mu\phi * \frac{\delta S^{(0)}[\phi]}{\delta\phi(x)} = \int dx\, \partial^\rho T_{\rho\mu} = 0. \tag{57}$$

We add an improvement term in order to get an improved energy-momentum tensor which is traceless for $m = 0$ [8]:

$$T^I_{\rho\mu} = T_{\rho\mu} + \frac{1}{6}(\eta_{\rho\mu}\Box - \partial_\rho\partial_\mu)(\phi * \phi). \tag{58}$$

The improvement term does not contribute to the divergence of the energy-momentum tensor which is given by

$$\partial^\rho T_{\rho\mu} = \partial^\rho T^I_{\rho\mu} = \frac{\lambda}{4!}[[\phi, \partial_\mu\phi]_M, \phi * \phi]_M \neq 0, \tag{59}$$

where we have introduced the Moyal bracket

$$[\phi_1(x), \phi_2(x)]_M := (\phi_1 * \phi_2)(x) - (\phi_2 * \phi_1)(x). \tag{60}$$

Result (59) is already given in [2].

From the above condition on σ, we also have $\sigma^{0i} = 0$ in the Minkowski case. Thus[4],

$$\int d^3x\, (\phi_1 * \phi_2 * \ldots * \phi_n)(x) =$$

$$= \int d^3x\, (\phi_n * \phi_1 * \ldots * \phi_{n-1})(x). \tag{61}$$

Eq. (59) implies

$$\int d^3x\, \partial^\rho T_{\rho\mu} = \partial^0 \int d^3x\, T_{0\mu} + \int d^3x \partial^i T_{i\mu} =$$

$$= \partial^0 \int d^3x\, T_{0\mu} = \int d^3x\, \frac{\lambda}{4!}[[\phi, \partial_\mu\phi]_M, \phi * \phi]_M = 0, \tag{62}$$

which means that there exists a conserved four momentum in this case:

$$\partial^0 P_\mu := \partial^0 \int d^3x\, T_{0\mu} = 0. \tag{63}$$

Additionally, $\sigma^{0i} = 0$ allows to establish unitarity [10].

[3] In [13], one finds some further useful remarks concerning translation symmetry in deformed quantum field theories.

[4] Note that the $\sigma_{\mu\nu}$-matrix is no longer invertible in general, and therefore we restrict our attention to the Moyal-deformed field theory.

As is well known, in the commutative case, the generators of the conformal group are given by moments of the energy-momentum tensor [8]. E.g., in the commutative case, the conserved current for dilatation symmetry is given by

$$D_\mu = x^\rho T^I_{\rho\mu}. \tag{64}$$

However, in the noncommutative case, one expects a breaking of the dilatation symmetry due to the fact that the energy-momentum tensor is not conserved. As a simple example, in the last section we study the broken dilatation symmetry in a Moyal—Weyl deformed field theory.

4. The Broken Dilatation Symmetry

In this section, we express the dilatation transformation in terms of a functional differential operator, i.e., we consider

$$W_D = \int dx\, \delta_D \phi * \frac{\delta}{\delta\phi(x)} = \int dx\, (1 + x^\mu * \partial_\mu)\phi * \frac{\delta}{\delta\phi(x)} \tag{65}$$

acting on the Minkowskian action $S^{(0)}[\phi]$ for a massless field given by

$$S^{(0)} = \int dx\, \left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{\lambda}{4!}(\phi)^4_*\right). \tag{66}$$

Using

$$x^\mu = (2\pi)^4 \int dp\, e^{ipx}\, i\frac{\partial}{\partial p_\mu}\delta^{(4)}(p),$$

$$\partial_\mu\phi(x) = \int dp\, e^{ipx}\, ip_\mu\tilde{\phi}(p), \tag{67}$$

one verifies, with the definition of the Moyal product (34),

$$x^\mu * \partial_\mu\phi(x) = x^\mu\partial_\mu\phi(x). \tag{68}$$

Then one gets, using the improved energy-momentum tensor (58),

$$W_D S^{(0)}[\phi] = -\int dx \Big[\partial^\rho \left(x^\mu * T^I_{\rho\mu}\right) +$$

$$+\frac{1}{2}(\phi * \Box\phi - \Box\phi * \phi) + \frac{1}{2}x^\mu * (\partial_\mu\phi * \Box\phi - \Box\phi * \partial_\mu\phi) +$$

$$+\frac{\lambda}{4!}x^\mu * \left(4\partial_\mu\phi * (\phi)^3_* - \partial_\mu(\phi)^4_*\right)\Big]. \tag{69}$$

It is straightforward to show that the terms involving the d'Alembertian in (69) vanish and thus one has

$$W_D S^{(0)}[\phi] = -\int dx \Big[\partial^\rho \left(x^\mu * T^I_{\rho\mu}\right) +$$

$$+\underbrace{\frac{\lambda}{4!}x^\mu * \left(4\partial_\mu\phi * (\phi)^3_* - \partial_\mu(\phi)^4_*\right)}_{=:B}\Big]. \tag{70}$$

A rather lengthy but straightforward calculation shows that the breaking B can be written as

$$B = -2\sigma^{\mu\nu}\frac{\partial S^{(0)}[\phi]}{\partial\sigma^{\mu\nu}} \tag{71}$$

which demonstrates that the breaking is determined by the deformation parameter $\sigma^{\mu\nu}$. Result (71) can be understood in the following way. An "infinitesimal" dilatation

$$\hat{x}'^\mu = (1 + \varepsilon)\hat{x}^\mu \qquad (\varepsilon \ll 1) \tag{72}$$

yields the following modified algebra for the operators $\hat{x}'^\mu$:

$$[\hat{x}'^\mu, \hat{x}'^\nu] = i(1 + 2\varepsilon)\sigma^{\mu\nu} + \mathcal{O}(\varepsilon^2). \tag{73}$$

This means that the change in the deformation parameter induced by infinitesimal dilatations is given by $\delta\sigma^{\mu\nu} = 2\sigma^{\mu\nu}$. Therefore one expects the following relation:

$$\int dx\, \delta_D\phi\frac{\delta S^{(0)}}{\delta\phi} + \delta\sigma^{\mu\nu}\frac{\partial S^{(0)}}{\partial\sigma^{\mu\nu}} = 0. \tag{74}$$

This reproduces exactly results (70), (71).

5. Conclusion and Outlook

In the previous sections, we have shown that one is able to construct an energy-momentum tensor which allows to define a conserved four momentum if $\sigma^{0i} = 0$. We have also demonstrated that the Noether theorem for translations exists already at the operator level in terms of the operators $\phi(\hat{x})$. Using the Moyal—Weyl correspondence between operators $\phi(\hat{x})$ and fields $\phi(x)$, we have also derived the energy-momentum tensor in the presence of a Moyal deformed interaction. Our result confirms the results of [1, 2].

ISSN 0503-1265. Укр. фіз. журн. 2002. Т. 47, N 3

In the last section, we have also considered the dilatation symmetry directly in a deformed field theory. We found that the Ward-identity of dilatation picks up a breaking proportional to the deformation parameter $\sigma^{\mu\nu}$. All our considerations are classical, i.e. without inclusion of radiative corrections. Our investigations may be the basis to study the trace anomaly at least at the one loop level. In a further work [12], we will try to give an answer whether the well-known trace anomaly [11] is modified in a Moyal—Weyl deformed scalar quantum field theory.

The work of J. M. G. is supported by the Danish Research Agency, the work of L. P. is supported in part by "Fonds zur Förderung der Wissenschaftlichen Forschung" (FWF) under contract P13125-PHY, and R. W. is a Marie-Curie Fellow.

1. *Krajewski T.* "Noncommutative geometry and fundamental interactions" (In French), arXiv:math-ph/9903047.
2. *Micu A.* // JHEP. – 2001. – **0001**. – P. 025; [arXiv:hep-th/0008057].
3. *Balazs N.L., Jennings B.K.* // Phys. Repts. – 1984. – **104**. – P. 347.
4. *Filk T.* // Phys. Lett. B. – 1996. – **376**. – P. 53.
5. *Ambjorn J., Makeenko Y.M., Nishimura J., Szabo R.J.* // JHEP. – 2000. – **0005**. – P. 023; [arXiv:hep-th/0004147].
6. *Madore J., Schraml S., Schupp P., Wess J.* // Eur. Phys. J. C. – 2000. – **16**. – P. 161; [arXiv:hep-th/0001203].
7. *Alvarez-Gaume L., Wadia S.R.* // Phys. Lett. B. – 2001. – **501**. – P. 319; [arXiv:hep-th/0006219].
8. *Coleman S.R., Jackiw R.*// Ann. phys. – 1971. – **67**. – P. 552.
9. *Jackiw R.* "Dilatation Symmetry and Light-cone commutators": Extended version of lectures delivered at the University of Turin, June, 1971; and at the DESY Summer School, July, 1971.
10. *Gomis J., Mehen T.* // Nucl. Phys. B. – 2000. – **591**. – P. 265; [arXiv:hep-th/0005129].
11. *Schweda M.*// Lett. al Nuovo cimento. – 1973. – **9**. – N 15. – P. 596-598.
12. In preparation.
13. *Pengpan T., Xiong X.* // Phys. Rev. D. – 2001. – **63**. – P. 085012; [arXiv:hep-th/0009070].

ТЕНЗОР ЕНЕРГІЇ-ІМПУЛЬСУ
У НЕКОМУТАТИВНИХ ПРОСТОРАХ —
ДЕЯКІ ПЕДАГОГІЧНІ КОМЕНТАРІ

А. Герхолд, Й. Грімштруп, Х. Гроссе, Л. Попп, М. Шведа, Р. Вулкенхаар

Р е з ю м е

Ми запрошуємо до дискусії про тензор енергії-імпульсу скалярної ϕ^4-теорії у некомутативному просторі. Формалізм Ньотер реалізований на оперативному рівні. Крім цього, розглянуто порушення симетрії масштабних перетворень у деформованій скалярній теорії поля Мойала — Вейля на класичному рівні.

ТЕНЗОР ЭНЕРГИИ-ИМПУЛЬСА
НА НЕКОММУТАТИВНЫХ ПРОСТРАНСТВАХ —
НЕКОТОРЫЕ ПЕДАГОГИЧЕСКИЕ КОММЕНТАРИИ

А. Герхолд, Й. Гримштруп, Х. Гроссе, Л. Попп, М. Шведа, Р. Вулкенхаар

Р е з ю м е

Мы приглашаем к дискуссии о тензоре энергии-импульса для скалярной ϕ^4-теории в некоммутативном пространстве. Формализм Э.Нетер реализован на операторном уровне. Кроме этого, рассматривается нарушение симметрии масштабных преобразований в деформированной скалярной теории поля Мойала — Вейля на классическом уровне.