APPLYING THE q-ALGEBRAS $U'_q(\mathrm{so}_n)$ TO QUANTUM GRAVITY: TOWARDS q-DEFORMED ANALOG OF SO(n) SPIN NETWORKS[1]

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UDC 538.9; 538.915; 517.957

Nonstandard q-deformed algebras $U'_q(\mathrm{so}_n)$, proposed a decade ago for the needs of representation theory, essentially differ from the standard Drinfeld—Jimbo quantum deformation of the algebras $U(\mathrm{so}_n)$ and possess with regard to the latter a number of important advantages. We discuss possible application of the q-algebras $U'_q(\mathrm{so}_n)$, within two different contexts of quantum/q-deformed gravity: one concerns q-deforming of D-dimensional ($D \geq 3$) euclidean gravity, the other applies to 2+1 anti-de Sitter quantum gravity (with space surface of genus g) in the approach of Nelson and Regge.

1. Introduction

Construction of quantum gravity belongs to most fundamental problems of modern quantum theory. During last decade and a half, new perspective tools for attacking and solving this problem have appeared among which we mention, first, the notion of spin networks (see e.g., [1]) closely connected with loop quantum gravity as well as with the so-called BF-type topological theories, and, second, the powerful methods of quantum groups and quantum algebras [2 — 5]. Our goal in this contribution is to consider potential applicability of the so-called nonstandard q-deformed algebras $U'_q(\mathrm{so}_n)$ introduced in [6] which are different from the standard (Drinfeld—Jimbo) quantum deformation [2, 3] of the Lie algebras of orthogonal groups, while possess a number of rather important advantages. Here we intend to make a preliminary steps towards extending the D-dimensional version of spin networks (more concretely, SO(D) simple spin networks) to the case of $U'_q(\mathrm{so}_n)$ related formulation. In the second part of our contribution, we briefly discuss the appearance of the $U'_q(\mathrm{so}_n)$ algebras in the context of anti-de Sitter 2+1 quantum gravity formulated with space-part being fixed as genus g Riemann surface so that $n = 2g + 2$.

2. Simple $G = \mathrm{SO}(n)$ Spin Networks

Let us first briefly dwell upon necessary setup concerning $G = \mathrm{SO}(n)$ spin networks.

A generalized spin network associated with a Lie group G, according e.g. to [7], is defined as a triple (Γ, ρ, I) where

Γ is an oriented graph, formed by directed edges and vertices;

ρ is a labeling of each edge e by an irreducible representation (irrep) ρ_e of G;

I is a labeling of each vertex v of Γ by an intertwinner I_v mapping tensor product of irreps incoming at v to the product of irreps outgoing from v.

Below, we are interested in the spin networks for the particular Lie group $G = \mathrm{SO}(n)$. Moreover, like in [7], we consider restricted case of $G = \mathrm{SO}(n)$ *simple* spin networks. Simple spin networks associated with $G = \mathrm{SO}(n)$ are evaluated as Feynman integrals over the coset space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, i.e. over the sphere S^{n-1}. Simplicity means that only the SO(n) representations of class 1 (with respect to SO($n-1$)) labeled by single nonnegative integer l, are employed.

Basic ingredient is the 'propagator' expressed in terms of zonal spherical functions $t^{nl}_{00}(y)$, $y = \cos\theta$, or, in view of the equality [8]

$$t^{Nl}_{00}(\cos\theta) = \frac{\Gamma(2p) l!}{\Gamma(2p+l)} C^p_l(\cos\theta) , \quad p = (N-2)/2 , \tag{1}$$

directly through the Gegenbauer polynomials:

$$G^{(N)}_m(x,y) = \frac{N+2m-2}{N-2} C^{(N-2)/2}_m(x \cdot y) . \tag{2}$$

Here the Gegenbauer polynomials $C^p_m(x)$, $l \geq 0$, satisfy the defining recursion relation

$$(l+1)C^p_{l+1}(x) =$$

[1]Presented at the XIIIth International Hutsulian Workshop "Methods of Theoretical and Mathematical Physics" (September 11 — 24 2000, Uzhgorod — Kyiv — Ivano-Frankivsk — Rakhiv, Ukraine) and dedicated to Prof. Dr. W. Kummer on the occasion of his 65th birthday.

$$2(p+l)xC_l^p(x) - (2p+l-1)C_{l-1}^p(x) \tag{3}$$

augmented with the initial value $C_0^p(x) = 1$, and obey the orthogonality relation

$$\int_{-1}^{1} (l+1)C_l^p(x)C_m^p(x)(1-x^2)^{p-\frac{1}{2}}\mathrm{d}x =$$

$$= \delta_{lm} \cdot \frac{\pi\Gamma(2p+l)}{2^{2p-1}l!(l+p)\Gamma^2(p)} \ . \tag{4}$$

Another important property is given by the linearization formula for the product of two Gegenbauer polynomials, namely

$$C_l^p(x)C_m^p(x) = \sum_{n=|l-m|}^{l+m} \frac{(n+p)\Gamma(n+1)\Gamma(g+2p)}{[\Gamma(p)]^2(g+p)!\Gamma(n+2p)} \times$$

$$\times \frac{\Gamma(g-l+p)\Gamma(g-m+p)\Gamma(g-n+p)}{\Gamma(g-l+1)\Gamma(g-m+1)\Gamma(g-n+1)} \ C_n^p(x) \tag{5}$$

where $g \equiv \frac{1}{2}(l+m+n)$ and the sum ranges over those values of n which are of the same evenness as $l+m+n$.

One of basic constructs in spin networks is the so-called Θ-graph whose evaluation is given by the expression

$$D(l,m,n;p) := \int_{-1}^{1} C_l^p(x)C_m^p(x)C_n^p(x)(1-x^2)^{p-\frac{1}{2}}\mathrm{d}x. \tag{6}$$

The result of evaluation is

$$D(l,m,n;p) = \frac{2^{1-2p}\pi}{[\Gamma(p)]^4} \frac{\Gamma(g+2p)}{\Gamma(g+p+1)} \times$$

$$\times \frac{\Gamma(g-m+p)\Gamma(g-n+p)}{\Gamma(g-l+1)\Gamma(g-m+1)\Gamma(g-n+1)} \ . \tag{7}$$

Using (6) one easily deduces the recurrence relation for the $D(l,m,n;p)$ in the form

$$\frac{l+1}{p+l}D(l+1,m,s;p) + \frac{2p+l-1}{p+l}D(l-1,m,s;p) =$$

$$= \frac{s+1}{p+s}D(l,m,s+1;p) + \frac{2p+s-1}{p+s}D(l,m,s-1;p) \tag{8}$$

(remark that in 4 dimensions all the multipliers become equal to 1).

Thus, Θ-graph $\theta^{(N)}(m_1,m_2,m_3)$ is nothing but integral of the product of three normalized propagators defined in (2):

$$\theta^{(N)}(m_1,m_2,m_3) =$$

$$= \frac{\Gamma(g+2p)\Gamma(p+1)}{\Gamma(g+p+1)\Gamma(2p)} \prod_{i=1}^{3} \frac{(m_i+p)\Gamma(g-m_i+p)}{\Gamma(p+1)\Gamma(g-m_i+1)}, \tag{9}$$

where $g = (m_1+m_2+m_3)/2$ is an integer, $g - m_i \geq 0$, $i = 1,2,3$, and $p = (N-2)/2$.

For $N = 4$,

$$\theta^{(4)}(m_1,m_2,m_3) = (m_1+1)(m_2+1)(m_3+1). \tag{10}$$

There exist a number of other results concerning (simple) SO(n) spin networks, for generic situation as well as for the particular cases of $n = 3,4$, which we however shall not discuss here further.

3. q-Deformed Analog of Spin Networks from q-ultraspherical Polynomials

To deal with q-deformed case we need some facts concerning q-ultraspherical polynomials. These are defined through the following recursion relations:

$$(1-q^n)C_n(x;\beta|q) = 2x(1-\beta q^{n-1})C_{n-1}(x;\beta|q)$$

$$-(1-\beta^2q^{n-2})C_{n-2}(x;\beta|q), \qquad (n \geq 2), \tag{11}$$

along with special values

$$C_0(x;\beta|q) = 1, \quad C_1(x;\beta|q) = 2(1-\beta)x/(1-q). \tag{12}$$

With $\beta = q^\lambda$, the "classical"limit $q \to 1$ yields

$$C_n(x;\beta|q) \xrightarrow{q\to 1} C_n^\lambda(x). \tag{13}$$

The explicit expression for the q-ultraspherical polynomials is [9] as follows:

$$C_n(x;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} \mathrm{e}^{\mathrm{i}(n-2k)\theta}$$

$$= \frac{(\beta;q)_n}{(q;q)_n}\mathrm{e}^{in\theta} {}_2\Phi_1(q^{-n},\beta;\beta^{-1}q^{1-n};q,q\beta^{-1}\mathrm{e}^{-2i\theta}). \tag{14}$$

In this formula, the notation $(a;q)_n$ means:

$$(a;q)_n = \begin{cases} 1, & n=0 \\ (1-a)(1-qa)...(1-q^{n-1}a), & n \geq 1. \end{cases} \tag{15}$$

It should be noted that it is also possible to present $C_n(x;\beta|q)$ in the form which employs basic hypergeometric function ${}_4\Phi_3$ or ${}_3\Phi_2$, see [9, 10].

Orthogonality relations for the q-ultraspherical polynomials are of principal importance. They are given by the relation

$$\int_0^\pi C_m(\cos\theta;\beta|q)C_n(\cos\theta;\beta|q)W_\beta(\cos\theta|q)\mathrm{d}\theta =$$

$$= \frac{\delta_{mn}}{h_n(\beta|q)}, \tag{16}$$

where the weight function and the normalization factor are as follows:

$$W_\beta(\cos\theta|q) = \frac{(\mathrm{e}^{2i\theta}, \mathrm{e}^{-2i\theta}; q)_\infty}{(\beta\mathrm{e}^{2i\theta}, \beta\mathrm{e}^{-2i\theta}; q)_\infty}, \tag{17}$$

$$h_n(\beta|q) = \frac{(q, \beta^2; q)_\infty (q;q)_n (1-\beta q^n)}{2\pi(\beta, \beta q; q)_\infty (\beta^2; q)_n (1-\beta)}, \tag{18}$$

with

$$(a_1, a_2; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \ ,$$

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k).$$

Linearization formula is another important fact about q-ultraspherical polynomials. It is given by the following Rogers' formula [9]:

$$C_m(x;\beta|q) C_n(x;\beta|q) =$$

$$= \sum_{k=0}^{\min(m,n)} A_{m,n,k}(\beta|q) C_{m+n-2k}(x;\beta|q) \tag{19}$$

with the notation

$$A_{m,n,k}(\beta|q) = \frac{(\beta;q)_{m-k}\ (\beta;q)_{n-k}\ (\beta;q)_k}{(q;q)_{m-k}\ (q;q)_{n-k}\ (q;q)_k} \times$$

$$\times \frac{(q;q)_{m+n-2k}\ (\beta^2;q)_{m+n-k}}{(\beta^2;q)_{m+n-2k}\ (\beta q;q)_{m+n-k}}\ \frac{1-\beta q^{m+n-2k}}{1-\beta}. \tag{20}$$

Now it is not hard to obtain the result for the q-deformed analog of Θ-graph (9), namely

$$D_q(m,n,s,\lambda) =$$

$$= \int_0^\pi C_m(x;\beta|q) C_n(x;\beta|q) C_s(x;\beta|q) W_\beta(\cos\theta|q) \mathrm{d}\theta =$$

$$= \frac{2\pi\ (\beta, \beta q; q)_\infty (\beta^2;q)_g (\beta;q)_{g-n} (\beta;q)_{g-m} (\beta;q)_{g-s}}{(q,\beta^2;q)_\infty (\beta q;q)_g (q;q)_{g-n} (q;q)_{g-m} (q;q)_{g-s}}, \tag{21}$$

where $m+n+s=2g,\ \ g \geq m,\ g \geq n,\ g \geq s$. Notice the obvious symmetry under exchanges: $m \leftrightarrow n \leftrightarrow s \leftrightarrow m$. Recursion relation for D_q is obtained in the form

$$\frac{1-q^{m+1}}{1-\beta q^m} D_q(m+1,n,s,\lambda) +$$

$$+ \frac{1-\beta^2 q^{m-1}}{1-\beta q^m} D_q(m-1,n,s,\lambda) =$$

$$= \frac{1-q^{s+1}}{1-\beta q^s} D_q(m,n,s+1,\lambda) +$$

$$+ \frac{1-\beta^2 q^{s-1}}{1-\beta q^s} D_q(m,n,s-1,\lambda). \tag{22}$$

Likewise, one can get evaluation for other particular (q-deformed analogs of) spin networks.

4. Covariance with Respect to q-algebras

Our main concern here is a possible relation of this stuff to quantum groups and/or q-algebras which correspond to the orthogonal Lie groups SO(n) and their corresponding Lie algebras. As it was shown by Sugitani in [11], zonal spherical functions associated with a particular realization S_q^N of quantum spheres are proportional to the q-ultrapsherical polynomials, that is

$$(q\text{-})\text{zonal spher. func.} \leftrightarrow C_k^{(N-2)/2}(Y;q^2)\ .$$

To this end, one starts with standard $U_q(\mathrm{so}_n)$ and constructs a q-analog of the coset SO(n)/SO($n-1$) by means of $U_q(\mathrm{so}_n)$ (corresponding to SO(n)) and a coideal (corresponding to SO($n-1$)):

$$J_q := \begin{cases} \left\langle e_2, ..., e_n, f_2, ..., f_n, \theta_1, \theta_2, \frac{q^{\epsilon_2}-1}{q-1}, ..., \frac{q^{\epsilon_n}-1}{q-1} \right\rangle, \\ \quad \text{for } B_n (n>1) \text{ and } \ D_n (n>2) \ \text{ series}, \\ \langle \theta_1 \rangle \quad\ \text{for } N=3 \\ \langle \theta_1, \theta_2, \frac{q^{\epsilon_2}-1}{q-1} \rangle \quad \text{for } \ N=4\ . \end{cases} \tag{23}$$

Here

$$\theta_1 := \begin{cases} s \cdot e_1 + (-1)^{n-1} t \cdot q^{1/2} q^{\epsilon_1} f_2 \cdots f_n f_n \cdots f_2 f_1 \\ \quad \text{for } B_n\ (n>1) \ \text{ series}, \\ s \cdot e_1 + (-1)^{n-2} t \cdot q^{\epsilon_1} f_2 \cdots f_{n-1} f_n f_{n-2} \cdots f_2 f_1 \\ \quad \text{for } D_n\ (n>2) \ \text{ series}, \\ s \cdot e_1 + t \cdot q^{1/2} q^{\epsilon_1} f_1 \quad (N=3), \\ s \cdot e_1 + t \cdot q^{\epsilon_1} f_2 \quad (N=4); \end{cases}$$

$$\tag{24}$$

$$\theta_2 := \begin{cases} t\cdot q^{1/2}q^{\epsilon_1}f_1 + (-1)^{n-1}s\cdot e_2\cdots e_n e_n\cdots e_2 e_1 \\ \quad \text{for } B_n \ (n>1) \ \text{ series}, \\ t\cdot q^{\epsilon_1}f_1 + (-1)^{n-2}s\cdot e_2\cdots e_{n-1}e_n e_{n-2}\cdots e_2 e_1 \\ \quad \text{for } D_n \ (n>2) \ \text{ series}, \\ t\cdot q^{\epsilon_1}f_1 + s\cdot e_2 \quad (N=4), \end{cases} \tag{25}$$

As follows from the results of [11], this left coideal subalgebra coincides with the nonstandard q-deformed algebra $U_q'(\mathrm{so}_n)$ from [6]. Note also that this same nonstandard (or twisted) q-deformed coideal subalgebra arises [12, 13] when one constructs a quantum analogue of the symmetric coset space $\mathrm{SU}(n)/\mathrm{SO}(n)$.

5. The q-algebra $U_q'(\mathrm{so}_n)$ (Bilinear Formulation)

Along with the definition in terms of trilinear relations originally given in [6], the q-algebra $U_q'(\mathrm{so}_n)$ may be equivalently defined in terms of *'bilinear' formulation.* To this end, the generators (set $k > l+1, \ \ 1 \le k,l \le n$)

$$I^{\pm}_{k,l} \equiv [I_{l+1,l}, I^{\pm}_{k,l+1}]_{q^{\pm 1}} \equiv$$

$$\equiv q^{\pm 1/2} I_{l+1,l} I^{\pm}_{k,l+1} - q^{\mp 1/2} I^{\pm}_{k,l+1} I_{l+1,l}$$

are introduced together with $I_{k+1,k} \equiv I^{+}_{k+1,k} \equiv I^{-}_{k+1,k}$. Then, the bilinear formulation of the q-algebra $U_q'(\mathrm{so}_n)$ reads:

$$[I^+_{lm}, I^+_{kl}]_q = I^+_{km}, \quad [I^+_{kl}, I^+_{km}]_q = I^+_{lm},$$

$$[I^+_{km}, I^+_{lm}]_q = I^+_{kl} \ \text{ if } k > l > m,$$

$$[I^+_{kl}, I^+_{mp}] = 0 \ \text{ if } k>l>m>p \ \text{ or } \ k>m>p>l; \tag{26}$$

$$[I^+_{kl}, I^+_{mp}] = (q-q^{-1})(I^+_{lp}I^+_{km} - I^+_{kp}I^+_{ml}) \ \text{ if } k>m>l>p.$$

Analogous set of relations exists which involves I^-_{kl} along with $q \to q^{-1}$ (denote this "dual" set by (26′)). In the 'classical' limit $q \to 1$, both (26) and (26′) reduce to those of so_n.

For instance, at $n = 3$, the q-algebra $U_q'(\mathrm{so}_3)$ is isomorphic [14] to the Fairlie – Odesskii algebra [15, 16] (recall that the q-commutator is defined as $[X,Y]_q \equiv q^{1/2}XY - q^{-1/2}YX$):

$$[I_{21}, I_{32}]_q = I^+_{31}, \ [I_{32}, I^+_{31}]_q = I_{21}, \ [I^+_{31}, I_{21}]_q = I_{32}; \tag{27}$$

at $n = 4$ the q-algebra $U_q'(\mathrm{so}_4)$ in addition involves:

$$\begin{array}{lll} [I_{32}, I_{43}]_q = I^+_{42}, & [I^+_{31}, I_{43}]_q = I^+_{41}, & [I_{21}, I^+_{42}]_q = I^+_{41}, \\ [I_{43}, I^+_{42}]_q = I_{32}, & [I_{43}, I^+_{41}]_q = I^+_{31}, & [I^+_{42}, I^+_{41}]_q = I_{21}, \\ [I^+_{42}, I_{32}]_q = I_{43}, & [I^+_{41}, I^+_{31}]_q = I_{43}, & [I^+_{41}, I_{21}]_q = I^+_{42}, \end{array}$$

$$[I_{43}, I_{21}] = 0, \ \ [I_{32}, I^+_{41}] = 0 \ ,$$

$$[I^+_{42}, I^+_{31}] = (q - q^{-1})(I_{21}I_{43} - I_{32}I^+_{41}).$$

The first relation in (27) is viewed as definition for third generator I^+_{31}; with this, the algebra is given in terms of q-commutators. Dual copy of $U_q'(\mathrm{so}_3)$ involves the generator $I^-_{31} = [I_{21}, I_{32}]_{q^{-1}}$ which enters the relations same as (27), but with $q \to q^{-1}$. Similar remarks apply to the generators I^+_{42}, I^+_{41}, as well as (dual copy of) the whole algebra $U_q'(\mathrm{so}_4)$.

6. Deformed Algebras $A(n)$ of Nelson and Regge

For (2 + 1)-dimensional gravity with cosmological constant $\Lambda < 0$, the Lagrangian density involves spin connection ω_{ab} and dreibein e^a, $a,b = 0,1,2$, combined in the $\mathrm{SO}(2,2)$-valued (anti-de Sitter) spin connection ω_{AB} of the form

$$\omega_{AB} = \begin{pmatrix} \omega_{ab} & \frac{1}{\alpha}e^a \\ -\frac{1}{\alpha}e^b & 0 \end{pmatrix},$$

and is given in the Chern—Simons (CS) form [17, 18]

$$\frac{\alpha}{8}(\mathrm{d}\omega^{AB} - \frac{2}{3}\omega^A_{\ F}\wedge\omega^{FB})\wedge\omega^{CD}\epsilon_{ABCD}.$$

Here $A,B = 0,1,2,3$, the metric is $\eta_{AB} = (-1,1,1,-1)$, and the CS coupling constant is connected with Λ, so that $\Lambda = -\frac{1}{3\alpha^2}$. The action is invariant under $\mathrm{SO}(2,2)$, leads to Poisson brackets and field equations. Their solutions, i.e. infinitesimal connections, describe space-time which is locally anti-de Sitter.

To describe global features of space-time, within fixed-time formulation, of principal importance are the *integrated connections* which provide a mapping S : $\pi_1(\Sigma) \to G$ of the homotopy group for a space surface Σ into the group $G = SL_+(2,R) \otimes SL_-(2,R)$ (spinorial covering of $\mathrm{SO}(2,2)$) and thoroughly studied in [19]. To generate the algebra of observables, one takes the traces

$$c^{\pm}(a) = c^{\pm}(a^{-1}) = \frac{1}{2}\mathrm{tr}[S^{\pm}(a)]$$

where

$$a \in \pi_1, \quad S^{\pm} \in SL_{\pm}(2,R).$$

For $g = 1$ (torus) surface Σ, the algebra of (independent) quantum observables has been derived [19], which turned out to be isomorphic to the cyclically symmetric Fairlie – Odesskii algebra [15, 16]. This latter algebra, however, is known to coincide [14] with the special $n = 3$ case of $U_q'(\mathrm{so}_n)$. So, natural question arises whether for surfaces of higher genera $g \geq 2$, the nonstandard q-algebras $U_q'(\mathrm{so}_n)$ also play a role.

Below, the positive answer to this question is given.

For the topology of spacetime $\Sigma \times \mathbf{R}$ (Σ being genus-g surface), the homotopy group $\pi_1(\Sigma)$ is most efficiently described in terms of $2g + 2 = n$ generators $t_1, t_2, \ldots, t_{2g+2}$ introduced in [20] and such that

$$t_1 t_3 \cdots t_{2g+1} = 1, \quad t_2 t_4, ..., t_{2g+2} = 1, \quad \prod_{i=1}^{2g+2} t_i = 1.$$

Classical gauge invariant trace elements ($n(n-1)/2$ in total) defined as

$$\alpha_{ij} = \frac{1}{2}\mathrm{Tr}(S(t_i t_{i+1} \cdots t_{j-1})), \qquad S \in SL(2,R), \tag{28}$$

generate concrete algebra with Poisson brackets, explicitly found in [20]. At the quantum level, to the algebra with generators (28) there corresponds quantum commutator algebra $A(n)$ specific for $2 + 1$ quantum gravity with negative Λ. For each quadruple of indices $\{j, l, k, m\}$, $j, l, k, m = 1, \ldots, n$, such that

$$i < j < m < k < i \ , \tag{29}$$

the algebra $A(n)$ of quantum observables reads [20]:

$$\begin{aligned} &[a_{mk}, a_{jl}] = [a_{mj}, a_{kl}] = 0, \\ &[a_{jk}, a_{kl}] = (1 - \tfrac{1}{K})(a_{jl} - a_{kl} a_{jk}), \\ &[a_{jk}, a_{km}] = (\tfrac{1}{K} - 1)(a_{jm} - a_{jk} a_{km}), \\ &[a_{jk}, a_{lm}] = (K - \tfrac{1}{K})(a_{jl} a_{km} - a_{kl} a_{jm}). \end{aligned} \tag{30}$$

Here the parameter K of deformation involves both α and Planck's constant, namely

$$K = \frac{4\alpha - ih}{4\alpha + ih}, \quad \alpha^2 = -\frac{1}{3\Lambda}, \quad \Lambda < 0. \tag{31}$$

Note that in (28) only one copy of the two $SL_{\pm}(2,R)$ is indicated. In conjunction with this, besides the deformed algebra $A(n)$ derived with, say, $SL_+(2,R)$ taken in (28) and given by (30), another identical copy of $A(n)$ (with the only replacement $K \to K^{-1}$) can also be obtained starting from $SL_-(2,R)$ taken in place of $SL(2,R)$ in (28). This another copy is independent from the original one: their generators mutually commute.

7. Isomorphism of the Algebras $A(n)$ and $U_q'(\mathrm{so}_n)$

To establish isomorphism [21] between the algebra $A(n)$ from (30) and the nonstandard q-deformed algebra $U_q'(\mathrm{so}_n)$ one has to make the following two steps.

— Redefine: $\{K^{1/2}(K-1)^{-1}\} a_{ik} \longrightarrow A_{ik}$,

— Identify: $A_{ik} \longrightarrow I_{ik}, \qquad K \longrightarrow q.$

Then, the Nelson—Regge algebra $A(n)$ is seen to translate exactly into the nonstandard q-deformed algebra $U_q'(\mathrm{so}_n)$ described above, see (26). We conclude that these two deformed algebras are isomorphic to each other (of course, for $K \neq 1$). Recall that n is linked to the genus g as $n = 2g+2$, while $K = (4\alpha - ih)/(4\alpha + ih)$ with $\alpha^2 = -\frac{1}{\Lambda}$.

Let us remark that it is the bilinear presentation (2) of the q-algebra $U_q'(\mathrm{so}_n)$ which makes possible establishing of this isomorphism. It should be stressed also that the algebra $A(n)$ plays the role of "intermediate" one: starting with it and reducing it appropriately, the algebra of quantum observables (gauge invariant global characteristics) is to be finally constructed. The role of Casimir operators in this process, as seen in [20], is of great importance. In this respect let us mention that the quadratic and higher Casimir elements of the q-algebra $U_q'(\mathrm{so}_n)$, for q being not a root of 1, are known in explicit form [13, 22] along with eigenvalues of their corresponding (representation) operators [22].

As it was shown in detail in [19], the deformed algebra for the case of genus $g = 1$ surfaces reduces to the desired algebra of three independent quantum observables which coincides with $A(3)$, the latter being isomorphic to the Fairlie — Odesskii algebra $U_q'(\mathrm{so}_3)$. The case of $g = 2$ is significantly more involved: here one has to derive, starting with the 15-generator algebra $A(6)$, the necessary algebra of 6 (independent) quantum observables. J.Nelson and T.Regge have succeeded [23] in constructing such an algebra. Their construction however is highly nonunique and, what is more essential, isn't seen to be efficiently extendable to general situation of $g \geq 3$.

Our goal in this note was to attract attention to the isomorphism of the deformed algebras $A(n)$ from [20] and the nonstandard q-deformed algebras $U_q'(\mathrm{so}_n)$ introduced in [6]). The hope is that, taking into account a significant amount of the already existing results concerning diverse aspects of $U_q'(\mathrm{so}_n)$ (the obtained various classes of irreducible representations [6, 14, 24—28] and others, as well as knowledge of Casimir operators and their eigenvalues depending on representations, etc.) we may expect for a further progress concerning construction of the desired algebra of $6g-6$ independent quantum observables for space surface of genus $g>2$.

The research described in this publication was made possible in part by Award No. UP1-2115 of the U.S. Civilian Research and Development Foundation (CRDF).

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ЗАСТОСУВАННЯ q-АЛГЕБР $U_q'(\mathrm{so}_n)$ ДО КВАНТОВОЇ ГРАВІТАЦІЇ: ПРО q-ДЕФОРМОВАНИЙ АНАЛОГ SO(n)-СПІНОВИХ СІТОК

О. М. Гаврилик

Р е з ю м е

Нестандартні q-деформовані алгебри $U_q'(\mathrm{so}_n)$, запропоновані десять років тому для потреб теорії представлень, істотно відрізняються від стандартної деформації Дрінфельда і Джімбо алгебр $U(\mathrm{so}_n)$ і мають перед останніми важливі переваги. Ми вивчаємо можливе застосування q-алгебр $U_q'(\mathrm{so}_n)$ у двох різних контекстах. Один з них стосується q-деформування D-вимірної ($D \geq 3$) евклідової гравітації на основі узагальнення формалізму спінових сіток, інший дає застосування до (2+1)-вимірної антидесіттерівської квантової гравітації (з рімановою поверхнею роду g) у підході Нельсона і Редже.

ПРИМЕНЕНИЕ q-АЛГЕБР $U_q'(\mathrm{so}_n)$ В КВАНТОВОЙ ГРАВИТАЦИИ: О q-ДЕФОРМИРОВАННОМ АНАЛОГЕ SO(n)-СПИНОВЫХ СЕТЕЙ

А. М. Гаврилик

Р е з ю м е

Нестандартные q-деформированные алгебры $U_q'(\mathrm{so}_n)$, предложенные десять лет тому назад в связи с потребностями теории представлений, существенно отличаются от стандартной деформации Дринфельда и Джимбо алгебр $U(\mathrm{so}_n)$ и обладают рядом преимуществ. Мы изучаем возможное применение q-алгебр $U_q'(\mathrm{so}_n)$ в двух различных контекстах. Один из них касается q-деформирования D-мерной ($D \geq 3$) эвклидовой гравитации на основе обобщения формализма спиновых сетей, другой — применения к (2+1)-мерной антидеситтеровской квантовой гравитации (с римановой поверхностью рода g) в подходе Нельсона и Редже.