EXACT INTERIOR SOLUTIONS FOR STATIC SPHERES IN THE EINSTEIN – CARTAN THEORY WITH TWO SOURCES OF TORSION

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In the framework of the problem of existence of exact interior solutions for static spherically symmetric configurations in the Einstein – Cartan theory (ECT), the distributions of perfect fluid and non-minimally coupled scalar field are considered. The exact solutions in the one-torsion ECT and two-torsion one are obtained. Some consequences of two sources of torsion are discussed.

Introduction

The aim of the present work is to obtain the analytic solutions of the Einstein–Cartan equations for static spherical configurations of matter.

Up till now the greatest number of exact interior solutions for the static spheres in ECT have been achieved for the models with a Weyssenhoff spinning fluid in the capacity of the source of torsion (see, for example, [1 – 7]).

Static spherically symmetric solutions with the Cosserat medium in Riemann–Cartan spacetime have been obtained in paper [8].

The exact interior solutions for charged spherical configurations in equlibrium, where the torsion has been generated by the vector field in ECT, have been achieved in [9].

In paper [10], the general exact solutions to the self–consistent set of the Einstein–Cartan equations for scalar static spheres, when the torsion is generated by the massless non-minimally coupled scalar field have been obtained.

Recently we have proposed a variant of the two-torsion ECT, in which a perfect fluid and a non-minimal coupling scalar field are the sources of torsion [11, 12]. In the present paper in the framework of this theory, we consider the static spherically symmetric distribution of perfect fluid and non-minimally coupled scalar field with a nonlinear potential.

1. The Lagrangian

The Lagrangian of the model is defined as

$$Ł = Ł_g + Ł_s + Ł_{fl} \ , \tag{1}$$

where $Ł_g$, $Ł_s$, $Ł_{fl}$ are the Lagrangians for the gravitational, non-minimally coupled scalar fields, perfect fluid, respectively.

The gravitation Lagrangian is chosen in the form [13, 14]

$$Ł_g = -R(\Gamma)/2æ \ , \tag{2}$$

where $R(\Gamma)$ is the curvature scalar obtained from the full connection, $æ = 8\pi G/c^4$ is the Einstein's constant.

For the connection coefficients, the expansion

$$\Gamma^k_{ij} = \{^k_{ij}\} + S_{ij\cdot}^{\ \ k} + S^k_{\cdot ij} + S^k_{\cdot ji} \tag{3}$$

is true, where $\{^k_{ij}\}$ are Christoffel symbols, $S_{ij\cdot}^{\ \ k} = \Gamma^k_{[ij]}$ is the torsion tensor which can be presented [14] as:

$$S_{ij\cdot}^{\ \ k} = \tilde{S}_{ij\cdot}^{\ \ k} + \frac{1}{3}(\delta^k_j S_i - \delta^k_i S_j) + \sqrt{-g}\varepsilon_{ij\cdot m}^{\ \ k}\check{S}^m \ . \tag{4}$$

Here, $\tilde{S}_{ij\cdot}^{\ \ k}$ is the traceless part of the torsion tensor, $S_i = S_{ik\cdot}^{\ \ k}$ is its trace and $\check{S}^m$ is its pseudotrace: $\check{S}^m = (1/6\sqrt{-g})\varepsilon^{mijk}S_{ijk}$.

The metric g_{ik} has signature $(-\ ,\ -\ ,\ -\ ,\ +)$, the Riemann and Ricci tensors are defined as

$$R_{ijk\cdot}^{\ \ \ \ m} = \Gamma^m_{jk,i} - \Gamma^m_{ik,j} + \Gamma^m_{ip}\Gamma^p_{jk} - \Gamma^m_{jp}\Gamma^p_{ik} ,$$

$$R_{jk} = R_{ijk\cdot}^{\ \ \ \ i} . \qquad (5)$$

The scalar field Lagrangian can be written in the form (see, for example, [11, 12, 15, 16]):

$$Ł_s = \frac{1}{4\pi}\Big\{\frac{\alpha_s}{2}[\Phi_{,k}\Phi^{,k} + \xi R(\Gamma)\Phi^2] - V(\Phi)\Big\} . \qquad (6)$$

Here, ξ is a coupling constant, $V(\Phi)$ is the potential of the scalar field; $\alpha_s = +1$ conforms to the material scalar field, $\alpha_s = -1$ corresponds to the "gravitational" scalar field.

Note that Lagrangian (6) is a covariant generalization of its counterpart in General Relativity (GR) and, for $\xi = 1/6$, we get the so-called conformal coupling (for $V(\Phi) = 0$ in the torsionless theory). As shown in [15], when $\alpha_s = -1$, $\xi = -1/6$, $V(\Phi) = -\frac{1}{2}m^2\Phi^2$, the scalar field corresponding to Lagrangian (6) is the axion field in GR.

The Lagrangian for a perfect fluid takes the form [11, 12, 17]:

$$Ł_{fl} = -\rho(c^2 + \Pi(\rho, e)) + k \overset{\Gamma}{\nabla}_i (\rho u^i) +$$

$$+k_1(u_i u^i - 1) + k_2 u^i \partial_i X + k_3 u^i \partial_i e . \qquad (7)$$

Here, ρ is the perfect fluid mass density; $\Pi(\rho, e)$ is its internal energy; k, k_1, k_2, k_3 are the Lagrange multipliers; X is the Lagrangian coordinates of matter particles; e is the entropy per volume; u^i is the four-velocity; $\overset{\Gamma}{\nabla}_i$ is the covariant derivative of the Riemann-Cartan space.

It follows from (7) that the torsion can interact with a perfect fluid only through its trace. In view of the fact that the analogous result has been derived [15] for the scalar field, the curvature scalar can be presented in the form:

$$R(\Gamma) = R(\{\}) + 4S^k_{;k} - \frac{8}{3}S_k S^k , \qquad (8)$$

where R({ }) is the Riemannian part of the curvature built from Christoffel symbols, the semicolon denotes a covariant derivative of the Riemannian space.

2. Field Equations and their Exact Solutions

The closed subsystem of equations in the framework of GR, which describes the gravitational interactions of a perfect fluid and a non-minimally coupled scalar field with the potential $V(\Phi)$, corresponding to Lagrangian (1), has the following form [11, 12]:

$$G_{ij}(\{\}) = æ(T^{fl}_{ij} + T^s_{ij}) + \Lambda_{ij} , \qquad (9)$$

$$(\Theta u^i)_{;i} = -(\varepsilon_{fl} + P_{fl}) , \qquad (10)$$

$$\Box\Phi - \xi\Phi R(\Gamma) + \alpha_s V' = 0 , \qquad (11)$$

where

$$T^{fl}_{ij} = (\varepsilon_{fl} + P_{fl})u_i u_j - P_{fl} g_{ij} , \qquad (12)$$

$$T^s_{ij} = \frac{\alpha_s}{4\pi}\Big\{\Phi_{,i}\Phi_{,j} - \frac{1}{2}\Big[\Phi_{,m}\Phi^{,m} + \xi R(\{\})\Phi^2 -$$

$$-2\alpha_s V(\Phi)\Big]g_{ij} + \xi\Big[-2S_i\nabla_j - 2S_j\nabla_i + 2g_{ij}S^n\nabla_n -$$

$$-\nabla_i\nabla_j + g_{ij}\Box + R_{ij}(\{\}) - \Lambda_{ij}\Big]\Phi^2\Big\} , \qquad (13)$$

$$\Lambda_{ij} = \frac{8}{3}S_i S_j - \frac{4}{3}S_k S^k g_{ij} , \qquad (14)$$

$$S^k = \frac{3}{2}\Psi(2\pi\alpha_s\Theta u^k + \xi\Phi\Phi^{,k}) , \qquad (15)$$

$$\varepsilon_{fl} = \rho(c^2 + \Pi(\rho, e)) , \qquad P_{fl} = \rho^2\partial\Pi/\partial\rho . \qquad (16)$$

Here, the ε_{fl} is the perfect fluid density; P_{fl} is its pressure; $\Box$ and ∇_i are the D'Alembertian operator and covariant derivative of the Riemannian space, respectively; $\Psi = æ(4\pi\alpha_s - \xi æ\Phi^2)^{-1}$; $\Theta = k\rho$; $V' = \partial V/\partial\Phi$.

The line element for the static spherically symmetric metric may be written in the curvature coordinates as

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2 d\Omega^2 + e^{\nu(r)}c^2 dt^2 , \qquad (17)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

As shown in [11], it follows from (10) for the static distribution of matter that

$$P_{fl} = -\varepsilon_{fl} = -C_1 , \quad C_1 = \text{const} , \quad C_1 > 0 . \qquad (18)$$

Eqs. (9) and (11) for metric (17) take the form:

$$e^{-\lambda}\Big(\frac{\lambda'}{r} - \frac{1}{r^2}\Big) + \frac{1}{r^2} = \Psi\Big\{4\pi\alpha_s(3\pi\alpha_s\Theta^2\Psi +$$

$$+C_1) + \alpha_s V(\Phi) - e^{-\lambda}\Big[2\xi\Phi\Phi'' + \xi\Phi\Phi'\Big(\frac{4}{r} - \lambda'\Big) +$$

$$+\Big(-\frac{1}{2}+2\xi+3\xi^2\Phi^2\Psi\Big)\Phi'^2\Big]\Big\}\ , \qquad (19)$$

$$e^{-\lambda}\Big(\frac{\nu'}{r}+\frac{1}{r^2}\Big)-\frac{1}{r^2}=\Psi\Big\{4\pi\alpha_s(3\pi\alpha_s\Theta^2\Psi-$$

$$-C_1)-\alpha_s V(\Phi)+e^{-\lambda}\Big[\Big(\frac{1}{2}-3\xi^2\Phi^2\Psi\Big)\Phi'^2+$$

$$+\xi\Phi\Phi'\Big(\frac{4}{r}+\nu'\Big)\Big]\Big\}\ , \qquad (20)$$

$$\frac{1}{2}e^{-\lambda}\Big(\nu''+\frac{\nu'^2}{2}+\frac{\nu'-\lambda'}{r}-\frac{\nu'\lambda'}{2}\Big)=\Psi\Big\{-4\pi\alpha_s(C_1-$$

$$-3\pi\alpha_s\Theta^2\Psi)-\alpha_s V(\Phi)+e^{-\lambda}\Big[2\xi\Phi\Phi''+\xi\Phi\Phi'\Big(\frac{2}{r}+$$

$$+\nu'-\lambda'\Big)+\Big(-\frac{1}{2}+2\xi+3\xi^2\Phi^2\Psi\Big)\Phi'^2\Big]\Big\}\ , \qquad (21)$$

$$e^{-\lambda}(1-6\xi^2\Phi^2\Psi)\Big[-\Phi''+\frac{1}{2}\Phi'\Big(\lambda'-\frac{4}{r}-\nu'\Big)\Big]-$$

$$-e^{-\lambda}\Big[\frac{2}{r^2}+\frac{2}{r}(\nu'-\lambda')+\nu''+\frac{\nu'^2}{2}-\frac{\nu'\lambda'}{2}\Big]\xi\Phi+$$

$$+\frac{2}{r^2}\xi\Phi+\alpha_s V'-12\pi\alpha_s\xi\Phi\Psi(\Theta u^k)_{;k}+$$

$$+24\pi\xi\Phi\Psi^2(\alpha_s\xi æ^{-1}e^{-\lambda}\Phi'^2+\pi\Theta^2)=0\ , \qquad (22)$$

where the prime denotes the differentiation with respect to r.

The scalar field potential $V(\Phi)$ is chosen in the form:

$$V(\Phi)=\beta\frac{\mu^2}{2}\Phi^2+\frac{\lambda}{4!}\Phi^4\ . \qquad (23)$$

Here, μ and λ are constants; $\beta=+1$ conforms to the massive scalar field, $\beta=-1$ corresponds to the Higgs type nonlinearity.

Note that the square of the trace of torsion

$$S^2=S_kS^k=\frac{9}{4}\Psi^2(4\pi^2\Theta^2-\xi^2\Phi^2\Phi'^2e^{-\lambda}) \qquad (24)$$

is directly proportional to the energy density of the torsion field ε_t:

$$\varepsilon_t=4S^2/3æ\ . \qquad (25)$$

To solve the field equations, let us proceed with the following assumptions:

(i) The Einstein's effective constant is positive

$$æ_{\rm eff}=\frac{æ}{1-\frac{\alpha_s\xi}{4\pi}æ\Phi^2}>0\ . \qquad (26)$$

(ii) The constants λ and μ are connected with the other parameters of the model by the following relations:

$$\lambda=\frac{6}{\pi}æ^2\xi^2C_1\ ,\qquad \mu^2=4æ|\xi|C_1\ . \qquad (27)$$

(iii) The following conditions must be fulfilled:

$$\varepsilon_{\rm eff}>0\ ,\qquad \Theta^2>0\ , \qquad (28)$$

where $\varepsilon_{\rm eff}$ is the effective energy density of the model.

It is not difficult to show that the set of Eqs. (19)–(22) admits the first integral:

$$\Phi'(|4\pi\alpha_s-æ\xi\Phi^2|)^{1/2}r^2\exp\Big(\frac{\nu-\lambda}{2}\Big)=C_2\ , \qquad (29)$$

where C_2 is an integration constant.

Let us consider some exact partial solutions.

1. Under the conditions $\alpha_s=\beta=-1$, $\xi<0$, the solution may be written as follows:

$$e^{\nu}=\frac{æ^3C_1^2C_2^2r^2}{(4\pi)^2}\ ,\qquad e^{-\lambda}=|\xi|\Big(\frac{r^2}{\hat r^2}-1\Big)\ ,$$

$$\Theta^2=\frac{2}{3æ^4C_1^2r^6}\ ,\qquad \Phi=n\sqrt{\frac{4\pi}{æ|\xi|}}\sqrt{1-\frac{\hat r^2}{r^2}}\ ,$$

$$\varepsilon_{\rm eff}=\frac{1}{æ}\Big(\frac{1+|\xi|}{r^2}-\frac{3|\xi|}{\hat r^2}\Big)\ , \qquad (30)$$

where $\hat r=(æC_1)^{-1/2}$, $n=\pm1$.

This solution exists for $|\xi|<1/2$ in the region $\hat r<r<\sqrt{\frac{1+|\xi|}{3|\xi|}}\hat r$. It is singular at $r=\hat r$ ($\Phi=0$, $e^{\lambda}\to\infty$). Note that the effective energy density $\varepsilon_{\rm eff}$ is finite at $r=\hat r$.

2. Under the condition $\alpha_s=-1$, $\beta=+1$, $\xi>0$, the solution is

$$e^{\nu}=\frac{æ^3C_1^2C_2^2r^2}{(4\pi)^2}\ ,\qquad e^{-\lambda}=\xi\Big(1-\frac{r^2}{\hat r^2}\Big)\ ,$$

$$\Theta^2=\frac{2}{3æ^4C_1^2r^6}\ ,\qquad \Phi=n\sqrt{\frac{4\pi}{æ\xi}}\sqrt{\frac{\hat r^2}{r^2}-1}\ ,$$

$$\varepsilon_{\rm eff}=\frac{1}{æ}\Big(\frac{1-\xi}{r^2}+\frac{3\xi}{\hat r^2}\Big)\ . \qquad (31)$$

For $0<\xi<1$, the solution exists in the region $0<r<\hat r$. It is singular at the center ($e^{\nu}\to0$, $\Phi\to\infty$, $S^2\to\infty$, $\varepsilon_{\rm eff}\to\infty$).

For $\xi\ge1$, solution (31) exists in the region $\sqrt{\frac{\xi-1}{3\xi}}\hat r<r<\hat r$. Note that the effective energy density is constant for $\xi=1$: $\varepsilon_{\rm eff}=3/æ\hat r^2$.This solution has a peculiarity at

$r = \hat{r}$: $\Phi = 0$, e^ν =const, $\varepsilon_{\rm eff} = (1+2\xi)/æ\hat{r}$, Θ =const, $e^\lambda \to \infty$.

It is interesting to observe that the energy density of the torsion field, corresponding to the solution (31),

$$\varepsilon_t = \frac{1}{2ær^2}\Big[1 - 6\xi\Big(1 - \frac{r^2}{\hat{r}^2}\Big)\Big] , \qquad (32)$$

is constant for $\xi = 1/6$: $\varepsilon_t = 1/2æ\hat{r}^2$.

3. Solutions for Static Spheres without Scalar Field

Now let us consider the one-torsion case without scalar field. Two exact partial solutions with the effective energy density $\varepsilon_{\rm eff}$ and pressure $P_{\rm eff}$

$$\varepsilon_{\rm eff} = C_1 + \frac{3}{4}æ\Theta^2 , \qquad P_{\rm eff} = -C_1 + \frac{3}{4}æ\Theta^2 , \qquad (33)$$

have been derived.

1. Under the condition $P_{\rm eff} = -\varepsilon_{\rm eff}/3$, we find a solution in the following form:

$$e^\nu = A = \text{const} , \qquad e^{-\lambda} = 1 - \frac{r^2}{2\hat{r}^2} ,$$

$$\Theta^2 = \frac{2C_1}{3æ} , \qquad \varepsilon_{\rm eff} = \frac{3}{2}C_1 . \qquad (34)$$

It follows from (34) that this solution exists in the region $0 \le r < \sqrt{2}\hat{r}$, it is free of singularities at the center and conforms to the case of constant torsion since $S^2 = 9æ^2\Theta^2/16$ =const.

2. In the case where the effective pressure and energy density are not bound up with a specific equation of state, the solution may be written as follows:

$$e^\nu = Br^2 , \qquad e^{-\lambda} = \frac{1}{2} - \frac{r^2}{3\hat{r}^2} ,$$

$$\Theta^2 = \frac{2}{3æ^2r^2} , \qquad \varepsilon_{\rm eff} = C_1 + \frac{1}{2ær^2} ,$$

$$P_{\rm eff} = -C_1 + \frac{1}{2ær^2} , \qquad (35)$$

where B is an integration constant ($B > 0$).

This solution exists in the region $0 < r < \sqrt{\frac{3}{2}}\hat{r}$. It is singular at the center ($e^\nu \to 0$, $S^2 \to \infty$, $\varepsilon_{\rm eff} \to \infty$, $P_{\rm eff} \to \infty$). It is easy to see that the contribution of torsion dominates at $r \to 0$. One can note that the function $P_{\rm eff}/\varepsilon_{\rm eff}$ equals unity at the center and decreases with increase of radius r.

Using the conditions of the matching in ECT [8]

$$[e^\nu] = 0 , \qquad [e^\lambda] = 0 , \qquad [P_{\rm eff}] = 0 , \qquad (36)$$

where $[f] \stackrel{def}{=} f_{out} - f_{in}$, with the Schwarzschild exterior metric

$$ds^2 = -\frac{dr^2}{1 - r_g/r} - r^2 d\Omega^2 + (1 - r_g/r)c^2dt^2 \qquad (37)$$

on the boundary of matter $r_b = \hat{r}/\sqrt{2}$, we obtain

$$B = (3r_b^2)^{-1} , \qquad r_g = \frac{2}{3}r_b . \qquad (38)$$

Conclusion

In this article, the exact partial static spherically symmetric solutions in the one-torsion ECT and the two-torsion one have been obtained.

In the one-torsion case where the torsion is generated by a perfect fluid with the vacuum equation of state, one of the solutions is non-singular at the center and conforms to the case of constant torsion. The other solution has the singularity at the center and admits the matching with the Schwarzschild exterior metric. Thus, the perfect fluid, which generates the torsion, may appear as the source of the Schwarzschild exterior solution.

In the two-torsion case, the exact solutions for the self-interacting massive scalar field ($\beta = +1$) and the self-interacting scalar field with the Higgs nonlinearity ($\beta = -1$) have been achieved. It is shown that the coupling constant ξ can influence the applicability of the solution.

Although the solution for $\beta = +1$ does not differ much in the two-torsion case from the second solution in the one-torsion one, it has its own characteristic features. Firstly, when the second source of torsion, induced by the non-minimal coupling scalar field, is taken into account, there is essentially restricted the region of the applicability of the solution ($\xi > 1$). Secondly, in the two-torsion case, the parameter Θ^2 changes with respect to r according to the law r^{-6}, as distinct from the law r^{-2} in the one-torsion one.

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ТОЧНІ ВНУТРІШНІ РОЗВ'ЯЗКИ ДЛЯ СТАТИЧНИХ СФЕР В ТЕОРІЇ ЕЙНШТЕЙНА—КАРТАНА З ДВОМА ДЖЕРЕЛАМИ КРУЧЕННЯ

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Р е з ю м е

У рамках проблеми існування точних внутрішніх розв'язків для статичних сферично-симетричних конфігурацій в теорії Ейнштейна—Картана (ТЕК) досліджуються розподіли ідеальної рідини і немінімально зв'язаного скалярного поля. Отримано розв'язки в одно- і двоторсійній ТЕК. Обговорюються деякі наслідки двох джерел кручення.

ТОЧНЫЕ ВНУТРЕННИЕ РЕШЕНИЯ ДЛЯ СТАТИЧЕСКИХ СФЕР В ТЕОРИИ ЭЙНШТЕЙНА—КАРТАНА С ДВУМЯ ИСТОЧНИКАМИ КРУЧЕНИЯ

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Р е з ю м е

В рамках проблемы существования точных внутренних решений для статических сферически-симметричных конфигураций в теории Эйнштейна—Картана (ТЭК) рассматриваются распределения идеальной жидкости и неминимально связанного скалярного поля. Найдены точные решения в одно- и двухторсионной ТЭК. Обсуждаются некоторые следствия двух источников кручения.