TOPOLOGICAL STRUCTURE OF MOMENTUM SPACE IN HIGH-ENERGY REGION

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Proceeding from the base notions of general topology, we show the existence of three different quantum field theories. Thus, the existence of two new wave mechanics beside the well-known Schrodinger one and non-closedness of the latter are a highly general mathematical fact. Here, we investigate the connection between the configuration and momentum spaces at high energies. Our consideration is limited by momentum spaces which are Euclidean in asymptotics (in the neighbourhood of the infinite distant point) and have some symmetry properties. In this case, both the symmetry group and harmonic analysis on such a space are very simple.

1. In quantum mechanics, there is a remarkable connection between small and large due to the Fourier transformation: namely, between small distances in the configuration space and large momenta in the momentum space expressed by the Heisenberg uncertainty relation, $\Delta X \Delta p \geq h/2$. But this connection is not quite right because, by making use of the L_2-space of functions on the momentum space A_p and the Lebesgue measure on it, we cannot advance into the high momentum region p. Here, the obstacle ultimately is the infinite Lebesgue measure of the full A_p. It follows from the analysis given in [1] that, in the region of high momenta (or energies), there is the proper wave mechanics and, in order to build a consistent field particle theory (free from ultraviolet divergences), the simple joint of the special relativity axioms (high-velocity mechanics) with quantum mechanical principles based on a separable Hilbert space is insufficient. Only at the classical level, high velocities and high energies mean the same. At the quantum level, they are quite different things.

The analysis in [1] showed that the cause of the above-mentioned difficulty hides in using a separable Hilbert space and differential topology on the configuration space in small. It turns out that, to deal with small distances (high energies), we have to refuse from the differential structure (the Newtonian model of space-time) and to consider the configuration space from the general topology point of view without any coordinate cover and differential topology. Here from this position, we investigate the connection between the configuration space A_X and the momentum space A_p at high energies.

2. In classical and quantum physics, space-time is considered to be a differential manifold. A differential structure is, strictly speaking, any supplementary structure on the space hiding the true nature of space. Indeed, to set a differential topology on A means first of all to cover it by any smooth coordinate system (see [2]). But the setting of such a system is not uniquely defined, not obligatory, and not always possible procedure [2] (physics does not depend on it ‾ a relativity principle). If A is not a smooth manifold and to set a differential topology is impossible, in this case, we have to consider A from the general topology point of view. To set such a topology means to consider some system of subsets of A. It is remarkable that such a system has any ring structure called the Boolean or Borel ring, see [3].

There are three and only three quite different non-equivalent topologies. They are:

α) the most weak topology connected with only two subsets: A itself and the empty set ϕ. The structure (ϕ, A) is a degenerated Boolean algebra (sometimes this non-Hausdorff topology is called as sticking together points; in this topology, A is the Bose condensate of its points, and it is compact due to the finite covering).

τ) Usual (natural or moderate) topology setting by a countable system of open submanifolds (if A satisfies the second countness axiom) $M \subset A$ (M can concide with A but never degenerates into the point). We say that in τ) (and α)) points of space obey to the Bose statistics.

ω) The most strong (or discrete) topology. In this topology, every connected neighbourhood of the point degenerates into this point, and A is a non-connected set of points or a discontinuum (a set without any topology or connectedness). Under such a topology, A is a non-separable space (if it is a non-discrete set).

We shall say that, in the topology ω), points of space obey to the Fermi statistics because there is only one point in each neighbourhood or there is not at all (empty set). It is important to note that topologies α) and ω) are unique. In the case τ), there are very many equivalent topologies (in particular, a differntial one), therefore, τ) is a widespread topology.

Further, we shall see (see also [1]) that, depending on physical conditions (state of a matter filling the space), the topology may change: with increase of the matter density (temperature), the topology will be stronger. We see the pure physical understanding of space differs from a mathematical axiomatization of it.

3. With each of these three topologies (Boolean rings), the following notions may be connected: i) proper measure (additive positive function on A) [3], ii) integral on A (the sum over this measure) [3], iii) proper class of measurable functions [3], and, if A admits a symmetry group (the so-called ergodic hypothesis), then iv) on A, there is the harmonic analysis ‾ Fourier transformation, mapping A into the dual (in the Pontryagin sense) space $\tilde{A}$ [4].

For τ), the measure is called Lebesgue (being invariant under the symmetry group, it is called a Haar measure). In this case, the natural class of functions (from the point of view of quantum mechanics) is $L_2(A)$. If A is a non-compact space with infinite Lebesgue measure, the dual space $\tilde{A}$ is non-compact too, and the latter has the same topology τ). If A is a compact space (finite Lebesgue measure), the dual one $\tilde{A}$ is discrete (the dual Pontryagin principle) [3].

In case α), we shall call the measure H.Bohr's one. The so-called H.Bohr compactification of A (denoted as bA [4]) is connected with such a measure. (The topology induced on A by the H.Bohr measure is called sometimes a Bohr-topology). Any compact (under a Lebesgue measure) submanifold $M \subset A$ ($M \neq A$) having a finite Lebesgue measure has a zero H.Bohr measure; the measure of all complete bA is equal to 1 (therefore, we say that the non-Hausdorff compact topology α) or Bohr-topology is in accordance with the H.Bohr measure and is not in accordance with the Lebesgue one). A natural class of functions on bA is the so-called class of almost periodic functions on A. The space $\tilde{A}$ dual to bA has a discrete topology ω) [5]. In this sense, topology ω) is dual to α) and vice versa. In ω), the measure of each isolated point is equal to 1.

The well-known topology τ) (a Lebesgue measure on both the configuration and momentum spaces) is used in the Schrodinger wave mechanics. Another two wave mechanics are connected with topologies ω) and α). We shall call the theory connected with ω) on the configuration space as high energy quantum mechanics. It is the field theory on the configuration discontinuum [1]. In this theory, the momentum space is a Bose-condensate of its points (see above). We call the theory connected with topology α) on the configuration space as low energy quantum mechanics.

4. Let us consider further only the high-energy quantum mechanics. We are interested in the structure of the momentum space A_p in this theory.

Let A_p is characterized by a symmetry group G (on such a space, the harmonic analysis may be developed [4]; we emphasize the harmonic analysis that we are going to consider on the momentum space). If G_0 is the stationary subgroup of a point $p \in A_p$, so A_p is a homogeneous space, i.e., it is a factor $G/G_0 = A_p$. For description of the high energy asymptotics of A_p, topology α) is well adopted (see [1]), so that the momentum space is the Bohr-compactification bA_p in this case (see above). All properties of bA_p are determined by the neighbourhood of the infinitely far point of A_p (which is not added to A_p), so that (or because) any finite part of A_p may be removed according to the Egorov theorem. Further, we consider only such spaces A_p which are asymptotically Euclidean $\sim \mathbf{R}_m$. In physics, momentum spaces are very often asymptotically Euclidean, for example, hyperboloids $\mathbf{p}^2 - p_0^2 = m^2$ in the Minkowski space. Harmonic analysis on $\mathbf{R}_m(p)$ is simple: it is the usual Fourier transformation $\int \psi(p)\, e^{-ipX}\, d^m p = \psi(X)$.

It is well known that the symmetry group of an affine space $\mathbf{R}_m$ is the Poincare group $E_m = \mathrm{SO}(m) \times)\, T_m$ (it is obtained by means of contraction of the exact symmetry group G of the space A_p). Thus, we consider $\mathrm{bA}_p \approx \mathrm{b}\mathbf{R}_m(p)$. All differences A_p from $\mathbf{R}_m(p)$ are connected with the finite part of A_p (its interior which is considered to be removed). A natural class of functions on $\mathrm{b}\mathbf{R}_m(p)$ is almost periodic functions on $\mathbf{R}_m(p)$ [5]. The peculiarity of bA_p as a space without interior is well seen in the spherical fibration $(\mathbf{R}_+, S^{m-1})$ of the space $\mathbf{R}_m$, where S^{m-1} is a sphere in $\mathbf{R}_m$. The analogous fibration for $\mathrm{b}\mathbf{R}_m$ is $(\mathbf{R}_+, \tilde{S}^{m-1})$, where $\tilde{S}^{m-1}$ is a 1-chain space over S^{m-1}. Indeed, if (infinite distant point is pricked) the interior of $\mathbf{R}_m$ is removed, then the rest part of $\mathbf{R}_m$ being connected is not linearly 1-connected. Further, we will see that a sphere without two points (nord and south poles) appears in such cases, for example, which has a covering space connected with $\tilde{S}^{m-1}$. It means that the symmetry group of $\mathrm{b}\,\mathbf{R}_m$ is $\widetilde{\mathrm{SO}}(m) \times)\, T_m$, where $\widetilde{\mathrm{SO}}(m)$ is a 1-chain group over $\mathrm{SO}(m)$ [6]. Obviously, $\tilde{S}^{m-1}$ is the factor $\widetilde{\mathrm{SO}}(m)/\widetilde{\mathrm{SO}}(2)$. Further, we consider only the case corresponding to $m = 3$ which is important in physical applications. Concerning the case $m = 2$, see [7] (usual quantum mechanics) and [8] (high energy quantum mechanics).

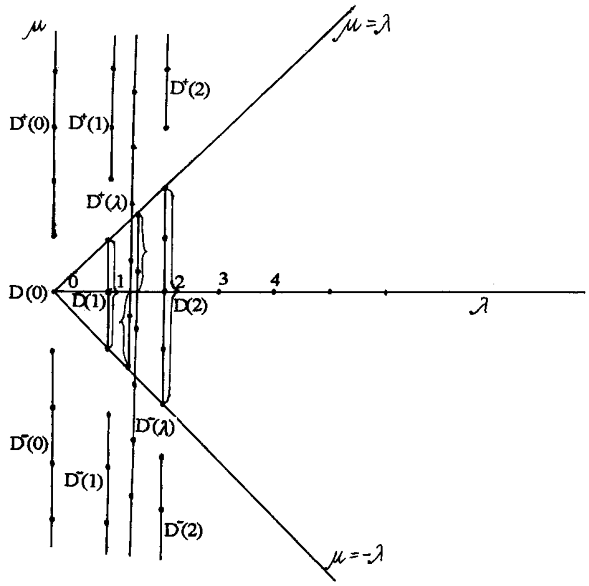
5. Let $p \in \mathbf{R}_3(p)$. It is known that the quantum mechanics on the momentum space $\mathbf{R}_3(p)$ is highly useful (see [9]) for description of spin properties of the kinematics and statistics of colliding particles (usualy, the scattering problem is considered in the configuration space).

If $\psi(p, \theta, \varphi)$ is a particle scattering amplitude considered on the fibration $(\mathbf{R}_+, S^2)$ of $\mathbf{R}_3(p)$, so the following formula holds true (the main formula of harmonic analysis on $(\mathbf{R}_+, S^2)$):

$$\psi(p, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} r^2 \, dr \, \psi_{rlm}(p, \theta, \varphi) \, \tilde{\psi}_{lm}(r), \quad (1)$$

where $\psi_{rlm}(p, \theta, \varphi) = R_{rl}(p) \, Y_m^{(l)}(\theta, \varphi)$, $Y_m^{(l)}(\theta, \varphi) = e^{im\varphi} P_l^m(\cos\theta)$ (P_l^m are normalized Lagrange polynomials) and $\{R_{rl}(p)\}$ is any complete (in separable Hilbert space) system of functions on $\mathbf{R}_+$, obeying, for example, the Laplace equation: $\Delta R_{rl}(p) = r^2 R_{rl}(p)$, where $\Delta = -\dfrac{\partial^2}{\partial p^2} - \dfrac{2}{p}\dfrac{\partial}{\partial p} + \dfrac{\mathbf{L}^2}{p^2}$ and $\mathbf{L}^2 = -(1 - X^2)\dfrac{\partial^2}{dX^2} + 2X\dfrac{\partial}{\partial X} - \dfrac{1}{1 - X^2}\dfrac{\partial^2}{\partial\varphi^2}$, $X = \cos\theta$. In such a case, $R_{rl}(p) = \dfrac{1}{\sqrt{p}} J_{l+\frac{1}{2}}(rp)$ see [7]. It is very important to emphasize that, in the high-energy quantum mechanics, only the momentum space remains to be a differential manifold endowed by the weak topology α) (in this case, the configuration space is a discontinuum, see above and [1]). In the limit $p \to \infty$, the z-axis in $\mathbf{R}_3(p)$, along which the particle momentum p is directed, must be subjected to a Bohr-compactification and hence be removed (see above). As a result, the rest of $b\mathbf{R}_3(p)$ is a linearly non 1-connected manifold, and, therefore, its spherical fibration is $(\mathbf{R}_+, \overline{S^2_{0,\pi}})$, where $S^2_{0,\pi}$ is a sphere with two pricked points corresponding to the azimuths $\theta = 0$ and $\theta = \pi$, and $\overline{S^2_{0,\pi}}$ is its universal covering space. Now, in the decomposition of a scattering amplitude $\psi(p, \theta, \varphi)$, the arbitrary angular momentum λ and magnetic quantum number μ (projection of λ on the quantization axis z) appear. Functions on a sphere corresponding them are connected with Gegenbauer polynomials well-adopted to the situation:

$$\overset{\pm}{f}_{\mu m}(\theta, \varphi) = A_{\mu m} \, e^{\pm i\mu\varphi} \frac{1}{(1 - X^2)^{\mu/2}} C_m^{1/2-\mu}(X),$$



Square integrable elements are denoted by brackets

$$m = 0, 1, 2, \ldots. \quad (2)$$

where $C_m^{\alpha}(X)$ are the Gegenbauer polynomials. $\overset{\pm}{f}_{\mu m}$ are the Cartan-Weyl basis of the irreducible infinite dimensional representation $\overset{\pm}{D}(\lambda)$ of SO(3) if and only if the quantum number μ is connected with λ by the condition $\mu = m - \lambda$ (see [10]). In this case, functions $\overset{\pm}{f}_{\mu m}$ are denoted as $\overset{\pm}{Y}{}_m^{(\lambda)}$ and satisfy the equations

$$-i\frac{\partial}{\partial\varphi}\overset{\pm}{Y}{}_m^{(\lambda)} = \pm\mu \overset{\pm}{Y}{}_m^{(\lambda)}, \quad \mathbf{L}^2 \overset{\pm}{Y}{}_m^{(\lambda)} = \lambda(\lambda + 1)\overset{\pm}{Y}{}_m^{(\lambda)}, \quad (3)$$

where $\mathbf{L}^2$ is the Casimir operator (see above). Systems $\{\overset{\pm}{Y}{}_m^{(\lambda)}\}$ form the basis of two irreducible representations $\overset{\pm}{D}(\lambda)$ which correspond to the same λ. $\overset{\pm}{D}(\lambda)$ are completely characterized by the lower $\overset{+}{Y}{}_0^{(\lambda)}$ and upper $\overset{-}{Y}{}_0^{(\lambda)}$ Cartan vectors [10]. Thus, the high-energy region is connected with an arbitrary angular momentum λ[1]. A representation with non-integer λ as if bifurcates:

[1] It is important to note that, in spite of using an arbitrary angular momentum in the suggested theory, the functions connected with it differ from those which are used in the Regge theory. The last is connected with Cauchy integral while the presented theory is connected with functional analysis.

the part of functions $\overset{+}{Y}{}_m^{(\lambda)}$, corresponding to the negative rates of $\mu = m - \lambda$ ($m = 0, 1, 2, \ldots, [\lambda]$, $[\lambda]$ is the entire part of λ) and square-integrable, belongs to the $\overset{+}{D}(\lambda)$-representation; the part of functions $\overset{-}{Y}{}_m^{(\lambda)}$, corresponding to the positive values of $\mu = \lambda - m$ ($m = 0, 1, 2, \ldots, [\lambda]$) and square-integrable, belongs to another representation $\overset{-}{D}(\lambda)$ (see Figure). All functions with $m = 0, 1, 2, \ldots, [\lambda]$ are square-integrable, but the functions $\overset{\pm}{Y}{}_m^{(\lambda)}$ with $m \geq [\lambda] + 1$ are not square-integrable: they have non-integrable singularities at $X = \pm 1$ (see (2)). Integrable functions $\overset{\pm}{Y}{}_m^{(\lambda)}$ form the orthonormalized system with respect to the H.Bohr scalar product (bmv means a Bohr mean value [1])

$$(\overset{+}{Y}{}_m^{(\lambda)}, \overset{+}{Y}{}_{m'}^{(\lambda')})' = \text{bmv}\int \overset{+}{Y}{}_m^{(\lambda)}(\theta, \varphi)\, \overline{\overset{+}{Y}{}_{m'}^{(\lambda')}(\theta, \varphi)} \times$$

$$\times\, d\varphi \sin\theta d\theta = \delta_{\mu\mu'}\delta_{mm'} = \delta_{mm'}\delta_{\lambda\lambda'} \tag{4}$$

(and the same for $\overset{-}{Y}{}_m^{(\lambda)}$, besides $(\overset{+}{Y}{}_m^{(\lambda)}, \overset{-}{Y}{}_{m'}^{(\lambda')}) = 0$) because

$$\lim_{\Phi\to\infty} \frac{1}{2\Phi}\int_{-\Phi}^{\Phi} e^{i(\mu-\mu')\varphi}\, d\varphi = \delta_{\mu\mu'}, \tag{5}$$

$$\int_{-1}^{1} \frac{dX}{(1-X^2)^{\mu}} C_m^{1/2-\mu}(X)\, C_{m'}^{1/2-\mu}(X) = \delta_{mm'}\frac{1}{A_{\mu m}^2},$$

where $A_{\mu m} = \dfrac{\Gamma(\frac{1}{2}-\mu)}{2^{\mu}}\sqrt{\dfrac{m!\,\Gamma(m-\mu+1/2)}{\pi\,\Gamma(m-2\mu+1)}}$. One can see that the functions $\overset{\pm}{Y}{}_m^{(\lambda)}$ with $m \leq [\lambda]$ resemble the functions $Y_m^{(l)}$ with the integer l belonging to the finite dimensional representation $D(l)$. Their local properties differ from those of $Y_m^{(l)}$ only near $\theta = 0$ and $\theta = \pi$ [10]. They form the uncountable basis of a non-separable Hilbert space with the scalar product (4). We have to consider the non-integrable functions $\overset{\pm}{Y}{}_m^{(\lambda)}$ with $m \geq [\lambda] + 1$ to be the generalized functions belonging to the rigged Hilbert space of almost periodic functions on $\overline{S^2_{0,\pi}}$ [2].

[2] It is necessary to note in accordance with [11] that the states corresponding to nonsquare-integrable functions are not realized in nature.

So any function $\psi(p, \theta, \varphi)$ on $(\mathbf{R}_+, \overline{S^2_{0,\pi}})$ may be represented in the form of Fourier series

$$\psi(p, \theta, \varphi) = \sum_{\lambda_i, m^{(i)}, r_j} \overset{\pm}{Y}{}_{m^{(i)}}^{(\lambda_i)}(\theta, \varphi)\, R_{r_j\lambda_i}(p)\, \tilde{\overset{\pm}{\psi}}_{\lambda_i, m^{(i)}}(r_j), \tag{6}$$

where $m^{(i)} \leq [\lambda_i]$ if $\psi(p, \theta, \varphi)$ belongs to the Hilbert space. Here, $R_{r\lambda}(p)$ satisfy the Laplace equation $\Delta R_{r\lambda}(p) = r^2 R_{r\lambda}(p)$ and are written as $R_{r\lambda}(p) = \frac{1}{\sqrt{p}} J_{\lambda+\frac{1}{2}}(pr)$ $(r \neq 0)$. These functions are normalized by the condition

$$\text{bmv}\int R_{r\lambda}(p)\, R_{r'\lambda}(p)\, p^2\, dp = \frac{1}{\pi r}\delta_{rr'}. \tag{7}$$

If $r = 0$, so $\lambda = 0$ and $R_{00}(p) = \frac{1}{p}$. In (6), the Fourier coefficients $\tilde{\overset{\pm}{\psi}}_{\lambda m}(r)$ are determined by the formula

$$\tilde{\overset{\pm}{\psi}}_{\lambda m}(r) = \text{bmv}\int \psi(p, \theta, \varphi)\, \overline{\overset{\pm}{Y}{}_m^{(\lambda)}(\theta, \varphi)\, R_{r\lambda}(p)}\, d\varphi \times$$

$$\times \sin\theta d\theta\, p^2\, dp = \sum_{\lambda_i, m^{(i)}, r_j \neq 0} \delta_{rr_j}\delta_{\lambda\lambda_i}\delta_{mm^{(i)}}\tilde{\overset{\pm}{\psi}}_{\lambda_i, m^{(i)}}(r_j) +$$

$$+\, \delta_{r0}\delta_{\lambda 0}\delta_{m0}\tilde{\overset{\pm}{\psi}}_{00}(0), \tag{8}$$

and the completeness condition is written as

$$\text{bmv}\int |\psi(p, \theta, \varphi)|^2 d\varphi \sin\theta\, d\theta\, p^2\, dp =$$

$$= \sum_{\lambda_i, m^{(i)}, r_j \neq 0} |\tilde{\overset{\pm}{\psi}}_{\lambda_i, m^{(i)}}(r_j)|^2 + |\tilde{\overset{\pm}{\psi}}_{00}(0)|^2. \tag{9}$$

6. Summarizing, we can say that when the 'center of weight" of particle physics shifts into the high-energy region, the measure on the momentum space A_p and its topology change radically: Bohr measure and non-Hausdorff topology should be considered on A_p. As a result, the linear connectedness of A_p changes. For all that, such a boundary condition arises that almost periodic functions on A_p and representations of the symmetry Lie group of A_p connected with them are used.

Although A_p remains a differential manifold, the dual configuration space A_X becomes a discontinuum (see [1]). The symmetry group of an isolated point of such a space is the 1-chain group $\widetilde{\text{SO}}(m)$. Indeed, it follows from [1] that the isolated point is, picturely speaking, a 'knot" of substance and accidence having a complicated dynamical structure described by the $\widetilde{\mathrm{SO}}(m)$ group.

Usually, one considers that if we draw near to the point, then its symmetry group does not change: it only depends on the dimension m of the space A_X to which the point belongs. However, it turns out that it is not so.

In conclusion, it is interesting to note that the theory of almost periodic (multivalued) representations of an arbitrary topological group G was built by von Neumann [12]. The necessary (but not sufficient) condition for existence of such representations is the noncompactness of G. However, soon after H.Weyl noticed [13] that $\mathrm{GL}(m)$ being a noncompact group has no such representations. But it turns out that $\widetilde{\mathrm{GL}}(m)$ connected with it (in any sense, covering GL [6]) has already almost periodic representations [14]. Here, the situation looks like that with two-valued spinor representations of $\mathrm{SO}(m)$.

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