# EFFECTIVE ACTION AT LOW-ENERGY IN QUANTUM ELECTRODYNAMICS

**V. P. GUSYNIN**

Bogolyubov Institute for Theoretical Physics, Nat. Acad. Sci. of Ukraine
*(14b, Metrolohichna Str., Kyiv 03143, Ukraine)*

**UDC 530.12;530145**
**‰ 2001**

The derivative expansion of the one-loop effective action in quantum electrodynamics (QED) is considered. The first term in such an expansion is the effective Heisenberg – Euler action for a constant electromagnetic field. An explicit expression for the next term containing two derivatives of the field strength $F_{\mu\nu}$, but exact in the magnitude of the field strength, is obtained. The correction to Schwinger's nonperturbative pair-production rate due to inhomogeneities in the background electric field is calculated.

**1.** The concept of low-energy or long-distance effective theory plays an important role in modern physics. In fact, all of known physics can be described by the effective low energy action of some fundamental theory (string theory or $M$-theory). The derivation of a low-energy effective action from some microscopic theory is an important task for various physical problems.

Perhaps, two most famous effective actions are the Ginzburg-Landau theory for the underlying BCS theory of superconductivity and the sigma model of the interaction between pions and nucleons. In the sixties, the so-called nonlinear chiral Lagrangians [1] were very popular, which describe the low-energy scattering processes of mesons and baryons rather well. Now we know the underlying microscopic theory: it is quantum chromodynamics (QCD), and the problem is to derive such chiral Lagrangians from QCD. Each of the effective theories is characterized by its own energy scale. For QCD, it is the $\Lambda$-parameter ($\sim$ 300 MeV). For energies much higher than $\Lambda$, the description in terms of quarks and gluons using the QCD Lagrangian looks simpler, but for energies much lower than $\Lambda$, the description in terms of quarks bound states ($\pi$, $\rho$, $K$, $N$) with chiral Lagrangians becomes simpler.

In this paper, we consider the low-energy quantum electrodynamics which, for energies below the electron mass, is described by an effective nonlinear theory for the electromagnetic field obtained after integrating out the fermion degrees of freedom. This problem is a rather old one and its roots go back to the well-known papers of Heisenberg and Euler [2] and Weisskopf [3]. Later, a progress was achieved by Schwinger [4] who, by using the proper time technique, rederived the one-loop effective action for the case of a constant electromagnetic field and analyzed the non-perturbative aspects of electron-positron pair production. In spite of the Schwinger Lagrangian has the virtue of being nonperturbative, taking into account the effects of the electromagnetic field to all orders, its validity is restricted to constant fields and, since that time, much effort has been devoted to finding a systematic way to generalize the result for strong, slowly varying fields [5] or fields localized in a finite region of space-time [6, 7].

Obviously, the most natural step deriving the low-energy effective action in quantum electrodynamics would be to take into account the effect of small deviations from the constant configuration of the field. In other words, the problem is to obtain an effective Lagrangian as an expansion in powers of derivatives of the field strength $F_{\mu\nu}$, but exact in the magnitude of the nonvanishing field strength:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_0 + \mathcal{L}_{\rm HE} + \partial_\lambda F_{\alpha\beta}\,\partial_\gamma F_{\sigma\delta}\,\mathcal{L}_1^{\lambda\alpha\beta\gamma\sigma\delta}(F_{\mu\nu}) + \ldots, \quad (1)$$

where, on right-hand side of Eq. (1), $\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}^2$, $\mathcal{L}_{\rm HE}$ is the famous Euler-Heisenberg Lagrangian for a constant field, and next terms are derivative corrections [1] (we will not address here the question of convergence of the perturbative derivative expansion in Eq. (1), for that see [7] where Borel's summation method was used for this purpose). Such a

[1] In [5] the heat-kernel method was applied to calculate the $\mathcal{L}_1$ term in the derivative expansion (1), however, the explicit result applicable to the most general case of the electromagnetic field was not given there.

generalization would be of great interest from the physical point of view, since it can be applied to studying instabilities (such as spontaneous symmetry breaking) in quantum field theory in an external field [8, 9], or be useful in treating heavy-ion scattering experiments, where strong varying electromagnetic fields are indispensible ingredients.

**2.** Let us start from the general formalism which was originally developed in [10, 11]. The generating functional of QED is written in terms of a path integral as

$$Z = \int d A_\mu \, d\psi \, d\bar\psi \, e^{i\int d^4x[-\frac{1}{4}F^2_{\mu\nu} + \bar\psi(i\hat{D} - m)\psi]}, \qquad (2)$$

and, after integration over the fermion field, we get

$$Z = \int d A_\mu \, e^{i\int d^4x\,(-\frac{1}{4}F^2_{\mu\nu}) + iW^{(1)}(A)} = $$
$$= \int d A_\mu \, e^{i\int d^4x\, \mathcal{L}_{\text{eff}}(A)}. \qquad (3)$$

Thus, the one-loop effective action $W^{(1)}(A)$ in QED reduces to computing the fermion determinant

$$W^{(1)}(A) = -i \ln \operatorname{Det}(i\hat{D} - m) =$$
$$= -\frac{i}{2}\ln \operatorname{Det}\left(D^2_\mu + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu} + m^2\right) =$$
$$= -\frac{i}{2}\int d^4x \operatorname{tr}\langle x|\ \ln\left(D^2_\mu + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu} + m^2\right)|\ x\rangle. \quad (4)$$

Here, $\hat{D} = \gamma^\mu D_\mu$, the covariant derivative is $D_\mu = \partial_\mu + ieA_\mu$, $\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$ and tr refers to spinor indices (we use the Minkowski metric $\eta_{\mu\nu} = (1, -1, -1, -1)$).

For slowly varying and small fields, the correction due to a fermion determinant is known to be

$$\mathcal{L}^{(1)}_{\text{eff}} = \frac{\alpha^2}{90\,m^4}\left[(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{4}(F_{\mu\nu}F^{*\mu\nu})^2\right] +$$
$$+ \frac{\alpha}{60\,\pi\,m^2}F_{\mu\nu}\,\Box\,F^{\mu\nu},\ F^{*\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}. \qquad (5)$$

For slowly varying but strong field, we want to calculate the effective Lagrangian as a derivative expansion around the nonvanishing field strength (1).

To derive the effective action, we employ a version of the so-called wordline (or string-inspired) formalism developed in [12 – 14]. Such an approach to the ordinary field theory, based on a path integral over one-dimensional world lines, was extended to the evaluation of Feynman diagrams for Green functions in higher loop orders [15 – 17]. For some recent applications of the worldline formalism as well as for an extensive list of references, see [18].

The effective Lagrangian can be represented through the diagonal matrix elements of the operator $U(\tau) = \exp(-i\tau H)$

$$\mathcal{L}^{(1)}(A) = \frac{i}{2}\int_0^\infty \frac{d\tau}{\tau}e^{-im^2\tau}\operatorname{tr}\langle x|\ \exp(-i\tau H)|\ x\rangle, \qquad (6)$$

where the second order differential operator $H$ is given by

$$H = -\Pi_\mu\Pi^\mu + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}(x), \qquad \Pi_\mu = iD_\mu. \qquad (7)$$

The matrix elements $\langle x|\ \exp(-i\tau H)|\ x\rangle$ entering the right-hand side of Eq. (6) may interpreted as the matrix elements of the evolution operator of a spinning particle with $\tau$ and $H$ being the proper time and the Hamiltonian of the particle. Following the standard approach [19], we represent the transition amplitude $\langle z|U(\tau)|y\rangle$ between points $x(0) = y$ and $x(\tau) = z$ in terms of a path integral over the real and Grassmann coordinates, $x_\mu(t)$ and $\psi_\mu(t)$, as

$$\operatorname{tr}\langle z|U(\tau)|y\rangle = \frac{1}{N}\int \mathcal{D}[x(t), \psi(t)] \times$$
$$\times \exp\left\{i\int_0^\tau dt\,[L_{\text{bos}}(x(t)) + L_{\text{fer}}(\psi(t), x(t))]\right\}, \qquad (8)$$

where $N$ is a normalization factor, and

$$L_{\text{bos}}(x) = -\frac{1}{4}\frac{dx_\nu}{dt}\frac{dx^\nu}{dt} - eA_\nu(x)\frac{dx^\nu}{dt},$$

$$L_{\text{fet}}(\psi, x) = \frac{i}{2}\psi_\nu\frac{d\psi^\nu}{dt} - ie\psi^\nu\psi^\lambda F_{\nu\lambda}(x). \qquad (9)$$

The integration in Eq. (8) goes over trajectories $x^\mu(t)$ and $\psi^\mu(t)$ parameterized by $t \in [0, \tau]$. The definition of integration measure assumes the following boundary conditions: $x(0) = y$, $x(\tau) = z$, $\psi(0) = -\psi(\tau)$. We choose a special gauge condition for the vector potential $A_\mu(x)$, namely the Fock – Schwinger gauge [20] $(x^\nu - y^\nu)A_\nu(x) = 0$, which leads to the series

$$A_\nu(x) = \sum_{n=0}^\infty \frac{(x^\lambda - y^\lambda)(x^{\nu_1} - y^{\nu_1})\ldots(x^{\nu_n} - y^{\nu_n})}{n!\,(n+2)} \times$$
$$\times\, \partial_{\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_n}F_{\lambda\nu}(y). \qquad (10)$$

This choice of the gauge for the vector potential turns out to be very convenient for developing a perturbative theory in the number of the derivatives of the electromagnetic field with respect to space-time coordinates.

Carrying out the change of the variable $x(t)$ for $x'(t) = x(t) - y$ in the path integral in Eq. (8) (henceforth, we omit the prime) and substituting Eq. (10) into Eq. (8), we obtain

$$\mathrm{tr}\,\langle z\,|\,U(\tau)\,|\,y\rangle = \frac{1}{N}\int D\,[x,(t),\psi(t)]\times$$
$$\times\exp\left[i\int_0^\tau dt\left(-\frac{1}{4}\frac{dx_\nu}{dt}\frac{dx^\nu}{dt} - \frac{e}{2}x^\lambda F_{\lambda\nu}(y)\frac{dx^\nu}{dt} +\right.\right.$$
$$\left.\left. + L^{\mathrm{int}}_{\mathrm{bos}}(x)\right)\right]\exp\left[i\int_0^\tau dt\left(\frac{i}{2}\psi_\nu\frac{d\psi^\nu}{dt} -\right.\right.$$
$$\left.\left. - ie\psi^\nu\psi^\lambda F_{\nu\lambda}(y) + L^{\mathrm{int}}_{\mathrm{fer}}(x,\psi)\right)\right]. \qquad (11)$$

The new boundary conditions for $x(t)$ are $x(0) = 0$ and $x(\tau) = z - y$. Notice that $F_{\mu\nu}$ in Eq. (11) does not depend on $x(t)$. As follows from Eq. (9) and (10), the expressions for the interacting terms, $L^{\mathrm{int}}_{\mathrm{bos}}(x)$ and $L^{\mathrm{int}}_{\mathrm{fer}}(x,\psi)$, containing the derivatives of $F_{\mu\nu}$ with respect to coordinates, take the form

$$L^{\mathrm{int}}_{\mathrm{bos}}(x) = \frac{e}{3}F_{\nu\lambda,\sigma}\frac{dx^\nu}{dt}x^\lambda x^\sigma +$$
$$+ \frac{e}{8}F_{\nu\lambda m\sigma\kappa}\frac{dx^\nu}{dt}x^\lambda x^\sigma x^\kappa + \ldots, \qquad (12)$$

$$L^{\mathrm{int}}_{\mathrm{fer}}(x,\psi) = -ieF_{\nu\lambda,\sigma}\psi^\nu\psi^\lambda x^\sigma -$$
$$-\frac{ie}{2}F_{\nu\lambda,\sigma\kappa}\psi^\nu\psi^\lambda x^\sigma x^\kappa + \ldots \qquad (13)$$

Now we see that the problem of obtaining the derivative expansion reduces to the evaluation of the path integral in Eq. (11) in the framework of perturbative theory with an infinite number of interacting terms given in Eqs. (12) and (13). Fortunately, for computing the effective action that includes only a finite number of derivatives, it is sufficient to consider only a finite number of interacting terms.

As usual, introducing real and Grassmann external sources, the matrix elements of the evolution operator can be represented as follows:

$$\mathrm{tr}\,\langle z\,|\,U(\tau)\,|\,y\rangle = \exp\left\{i\int_0^\tau dt\left[L^{\mathrm{int}}_{\mathrm{bos}}\left(\frac{1}{i}\frac{\delta}{\delta\eta(t)}\right) +\right.\right.$$
$$\left.\left. + L^{\mathrm{int}}_{\mathrm{fer}}\left(\frac{1}{i}\frac{\delta}{\delta\eta(t)}, -\frac{\delta}{\delta\xi(t)}\right)\right]\right\}Z_\tau[\eta,\xi](z;y)\Big|_{\eta=0,\,\xi=0}, \qquad (14)$$

where the generating functional is just the Gaussian path integral,

$$Z_\tau[\eta,\xi]\,(z;y) = \frac{1}{N}\int D\,[x(t),\psi(t)]\times$$
$$\times\exp\left[\frac{i}{2}\int_0^\tau dt\left(-\frac{1}{2}\frac{dx_\nu}{dt}\frac{dx^\nu}{dt} - ex^\lambda F_{\lambda\nu}(y)\frac{dx^\nu}{dt} +\right.\right.$$
$$\left.\left. + 2\eta_\nu x^\nu\right)\right]\exp\left[-\frac{1}{2}\int_0^\tau dt\left(\psi_\nu\frac{d\psi^\nu}{dt} -\right.\right.$$
$$\left.\left. - 2e\psi^\nu\psi^\lambda F_{\nu\lambda}(y) + 2\xi_\nu\psi^\nu\right)\right]. \qquad (15)$$

The calculation of this generating functional reduces to finding the 'classical" trajectories for $x_\nu(t)$ and $\psi_\nu(t)$ satisfying the appropriate boundary conditions, and to computing the determinants of the one-dimensional differential operators,

$$O_1 = \frac{\eta_{\mu\nu}}{2}\frac{d^2}{dt^2} - eF_{\mu\nu}\frac{d}{dt}, \quad \text{and} \quad O_2 = i\eta_{\mu\nu}\frac{d}{dt} - 2ieF_{\mu\nu}, \qquad (16)$$

defined on the interval $[0,\tau]$ with the periodic and antiperiodic boundary conditions for their eigenstates, respectively. The result of the path integration in Eq. (15) in the case of the coincident arguments $z = y = x$ reads

$$Z_\tau[\eta,\xi]\,(x;x) = C_0\sqrt{\frac{\mathrm{Det}\,(O_2)}{\mathrm{Det}'\,(O_1)}}\exp\left(\frac{i}{2}S^{\mathrm{bos}}_{\mathrm{cl}}[\eta] -\right.$$
$$\left. - \frac{1}{2}S^{\mathrm{fer}}_{\mathrm{cl}}[\xi]\right), \qquad (17)$$

where the normalization constant $C_0$ should be determined by comparing the result with the Schwinger one, or by satisfying the normalization condition $Z_{\tau=0}[\eta,\xi]\,(z;y) = \delta(z-y)$: $C_0 = -i/(2\pi\tau)^2$. The prime in Eq. (17) denotes skipping a zero mode in the definition of the determinant.

The expressions for $S^{\mathrm{bos}}_{\mathrm{cl}}$ and $S^{\mathrm{fet}}_{\mathrm{cl}}$ are quadratic forms in the external sources

$$S^{\mathrm{bos}}_{\mathrm{cl}}[\eta] = \int_0^\tau dt_1\int_0^\tau dt_2\,\eta_\nu(t_1)\,D^\nu_\lambda(t_1,t_2)\,\eta^\lambda(t_2),$$

$$S_{\rm cl}^{\rm fer}[\xi] = \int_0^{\tau} dt_1 \int_0^{\tau} dt_2\, \xi_\nu(t_1)\, S^{\nu}_{\ \lambda}(t_1, t_2)\, \xi^\lambda(t_2), \qquad (18)$$

where the Green functions are given in terms of functions of the matrix argument $F_{\mu\nu}$

$$D(t_1, t_2) = \frac{1}{2eF}\Big[\hat{\text{U}}(t_1 - t_2)\,(1 - e^{2eF(t_1 - t_2)}) + $$
$$+ \coth(eF\tau)\,(1 + e^{2eF(t_1 - t_2)}) - $$
$$- \frac{e^{eF(\tau - 2t_2)} + e^{eF(2t_1 - \tau)}}{\sinh(eF\tau)}\Big], \qquad (19)$$

$$S(t_1, t_2) = \frac{1}{2}\,[\hat{\text{U}}(t_1 - t_2) - \tanh(eF\tau)]\, e^{2eF(t_1 - t_2)}. \qquad (20)$$

Substitution of Eqs. (17), (19) and (20) into Eq. (14) leads to the expression for $\mathrm{tr}\langle x\,|\,U\,|\,x\rangle$. After expanding the exponent in powers of the operator-valued interacting terms, $L_{\rm bos}^{\rm int}$ and $L_{\rm fer}^{\rm int}$ (containing the functional derivatives with respect to the sources $\eta_\mu(t)$ and $\xi_\mu(t)$), one has to calculate the result of the derivative action on the generating functional. Starting from this point, we have to restrict ourselves to a specific finite number of derivatives in the effective action. For terms up to two derivatives, we obtain

$$\mathrm{tr}\langle x\,|\,U(\tau)\,|\,x\rangle = C_0\sqrt{\frac{\mathrm{Det}\,(O_2)}{\mathrm{Det}'(O_1)}}\;\times$$
$$\times\Big\{1 - \frac{i}{8}\,e\,F_{\nu\lambda,\,\mu\kappa}\int_0^{\tau} dt\,[\dot{D}^{\nu\lambda}(t,t)\,D^{\mu\kappa}(t,t) + \dot{D}^{\nu\mu}(t,t)\times$$
$$\times D^{\lambda\kappa}(t,t) + \dot{D}^{\nu\kappa}(t,t)\,D^{\lambda\mu}(t,t) + 4\,S^{\nu\lambda}(t,t)\,D^{\mu\kappa}(t,t)] -$$
$$- \frac{i}{18}\,e^2 F_{\nu\lambda,\,\mu}F_{\sigma\kappa,\,\rho}\int_0^{\tau}\!\!\int_0^{\tau} dt_1\,dt_2\Big[9D^{\mu\rho}(1,2)\times$$
$$\times\,(S^{\kappa\sigma}(2,2)\,S^{\lambda\nu}(1,1) - 2S^{\kappa\lambda}(2,1)\,S^{\sigma\nu}(2,1)) +$$
$$+\,6S^{\sigma\kappa}(2,2)\,(\dot{D}^{\nu\lambda}(1,1)\,D^{\mu\rho}(1,2) + \dot{D}^{\nu\mu}(1,1)\times$$
$$\times\,D^{\lambda\rho}(1,2) + \dot{D}^{\nu\rho}(1,2)\,D^{\lambda\mu}(1,1)) + \dot{D}^{\nu\lambda}(1,1)\times$$
$$\times\,\dot{D}^{\sigma\kappa}(2,2)\,D^{\mu\rho}(1,2) + 2\dot{D}^{\nu\lambda}(1,1)\,(\dot{D}^{\sigma\rho}(2,2)\times$$
$$\times\,D^{\mu\kappa}(1,2) + \dot{D}^{\sigma\mu}(2,1)\,D^{\kappa\rho}(2,2)) +$$
$$+\,\dot{D}^{\nu\mu}(1,1)\,\dot{D}^{\sigma\rho}(2,2)\,D^{\lambda\kappa}(1,2) +$$
$$+\,2\,\dot{D}^{\nu\kappa}(1,2)\,(\dot{D}^{\sigma\rho}(2,2)\,D^{\lambda\mu}(1,1) +$$
$$+\,\dot{D}^{\sigma\mu}(2,1)\,D^{\lambda\rho}(1,2)) + \dot{D}^{\nu\kappa}(1,2)\times$$
$$\times\,\dot{D}^{\sigma\lambda}(2,1)\,D^{\mu\rho}(1,2) + \dot{D}^{\nu\rho}(1,2)\,\dot{D}^{\sigma\mu}(2,1)\times$$
$$\times\,D^{\lambda\kappa}(1,2) + \ddot{D}^{\nu\sigma}(1,2)\,(D^{\lambda\mu}(1,1)\,D^{\kappa\rho}(2,2) +$$
$$+\,D^{\lambda\kappa}(1,2)\,D^{\mu\rho}(1,2) + D^{\lambda\rho}(1,2)\,D^{\mu\kappa}(1,2))\Big]\Big\}. \qquad (21)$$

Here, the dotted functions are defined by the expressions

$$\dot{D}^{\mu\nu}(1,2) \stackrel{\rm def}{=} \frac{\partial}{\partial t_1}D^{\mu\nu}(t_1, t_2),$$

$$\ddot{D}^{\mu\nu}(1,2) \stackrel{\rm def}{=} \frac{\partial^2}{\partial t_1 \partial t_2}D^{\mu\nu}(t_1, t_2),$$

$$\dot{D}^{\mu\nu}(t,t) \stackrel{\rm def}{=} \lim_{t_0\to t}\frac{\partial}{\partial t_0}D^{\mu\nu}(t_0, t). \qquad (22)$$

Having representation (21) together with the Green functions (19) and (20), one is left with a need to perform the integrations over the proper time. For that, it is convenient to use the set of matrices [21] $A^{\nu\lambda}_{(j)}$ with $j \in \{1, 2, 3, 4\}$,

$$A_{(j)\,\mu\nu} = \frac{-\bar{f}_j^{\,2}\,\eta_{\mu\nu} + f_j F_{\mu\nu} + F^{\,2}_{\mu\nu} - i\bar{f}_j \overset{*}{F}_{\mu\nu}}{2\,(f_j^2 - \bar{f}_j^2)}, \qquad (23)$$

which are eigenvalue matrices of the field strength tensor $F^{\mu\nu}$:

$$F^{\nu\lambda}A_{(i)\,\lambda\mu} = A^{\nu\kappa}_{(i)}F_{\kappa\mu} = f_i\,A^{\nu}_{(i)\,\mu}. \qquad (24)$$

Here, $f_{1,2} = \pm\, iK_-$, $f_{3,4} = \pm\, K_+$, $\bar{f}_{1,2} = \mp\, K_+$, $\bar{f}_{3,4} = \mp\, iK_-$ and

$$K_+ = \sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}},\quad K_- = \sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}},$$
$$\mathcal{F} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu},\qquad \mathcal{G} = \frac{1}{8}\hat{\text{U}}^{\mu\nu\lambda\kappa}F_{\lambda\kappa}F_{\mu\nu}. \qquad (25)$$

Other useful properties of these matrices that will be used below are

$$\sum_j A^{\mu\nu}_{(j)} = \eta^{\mu\nu},\quad A^{\mu}_{(j)\,\mu} = 1,\quad A^{\mu\nu}_{(k)}A_{(j)\,\nu\lambda} = \delta_{kj}A^{\mu}_{(j)\,\lambda}. \qquad (26)$$

i.e., the matrices $A^{\mu\nu}_{(j)}$ are projectors and, for any function $\Phi(F)$ of the tensor argument $F_{\mu\nu}$, we can write

$$\Phi(F)_{\mu\nu} = \sum_j A_{(j)\,\mu\nu}\,\Phi(f_{(j)}). \tag{27}$$

The straightforward though tedious computation gives the result for the diagonal matrix elements of $U(\tau)$:

$$\mathrm{tr}\,\langle x\,|\,U(\tau)\,|\,x\rangle = \mathrm{tr}\,\langle x\,|\,U(\tau)\,|\,x\rangle_0 =$$

$$= \Bigg[1 - \frac{i}{8}eF_{\nu\lambda,\,\mu\kappa}\sum_{j,\,l}(C^V(f_j, f_l)\,(A^{\nu\lambda}_{(j)}A^{\mu\kappa}_{(l)} + 2A^{\nu\mu}_{(j)}A^{\lambda\kappa}_{(l)}) + 2C^W(f_i, f_l)\,A^{\lambda\nu}_{(j)}A^{\mu\kappa}_{(l)}) -$$

$$- \frac{i}{18}e^2F_{\nu\lambda,\,\mu}F_{\sigma\kappa,\,\rho}\sum_{j,\,l,\,k}\Big(9\,C_1^{WW}(f_j, f_l, f_k)\times A^{\kappa\sigma}_{(j)}A^{\lambda\nu}_{(l)}A^{\mu\rho}_{(k)} + 9\,C_2^{WW}(f_i, f_l, f_k)\,A^{\kappa\lambda}_{(j)}A^{\sigma\nu}_{(l)}A^{\mu\rho}_{(k)} +$$

$$+ 6\,C_1^{VW}(f_j, f_l, f_k)\,A^{\sigma\kappa}_{(j)}(A^{\nu\lambda}_{(l)}A^{\mu\rho}_{(k)} + A^{\nu\mu}_{(l)}A^{\lambda\rho}_{(k)}) +$$

$$+ 6\,C_2^{VW}(f_j, f_l, f_k)\,A^{\sigma\kappa}_{(j)}A^{\nu\rho}_{(l)}A^{\lambda\mu}_{(k)} - C_1^{VV}(f_j, f_l, f_k)\times$$

$$\times(A^{\nu\lambda}_{(j)}A^{\kappa\sigma}_{(l)}A^{\mu\rho}_{(k)} + A^{\nu\mu}_{(j)}A^{\kappa\rho}_{(l)}A^{\lambda\sigma}_{(k)} + 2A^{\nu\lambda}_{(j)}A^{\kappa\rho}_{(l)}A^{\mu\sigma}_{(k)}) -$$

$$- C_2^{VV}(f_j, f_l, f_k)\,(A^{\nu\sigma}_{(j)}A^{\kappa\lambda}_{(l)}A^{\mu\rho}_{(k)} + A^{\nu\rho}_{(j)}A^{\kappa\mu}_{(l)}A^{\lambda\sigma}_{(k)} +$$

$$+ 2A^{\nu\sigma}_{(j)}A^{\kappa\mu}_{(l)}A^{\lambda\sigma}_{(k)}) - 2\,C_3^{VV}(f_j, f_l, f_k)\,(A^{\nu\lambda}_{(j)}A^{\kappa\mu}_{(l)}A^{\sigma\rho}_{(k)} +$$

$$+ A^{\kappa\rho}_{(j)}A^{\nu\sigma}_{(l)}A^{\lambda\mu}_{(k)}) - C_4^{VV}(f_j, f_l, f_k)\,A^{\nu\kappa}_{(j)}A^{\lambda\mu}_{(l)}A^{\sigma\rho}_{(k)} -$$

$$- C_5^{VV}(f_j, f_l, f_k)\,A^{\nu\kappa}_{(j)}(A^{\lambda\sigma}_{(l)}A^{\mu\rho}_{(k)} + A^{\lambda\rho}_{(l)}A^{\mu\sigma}_{(k)})\Big)\Bigg], \tag{28}$$

where the explicit expressions for the coefficients $C_i^{XY}(\alpha, \beta, \gamma)$ are given in [11] and the diagonal matrix elements $\mathrm{tr}\,\langle x\,|\,U(\tau)\,|\,x\rangle_0$ correspond to the non-derivative case [4],

$$\mathrm{tr}\,\langle x\,|\,U(\tau)\,|\,x\rangle_0 = -\frac{i}{4\pi^2\tau^2}(e\tau K_-)(e\tau K_+)\times$$

$$\times\cot(e\tau K_-)\coth(e\tau K_+). \tag{29}$$

In case of a pure magnetic field ($B \neq 0$, $E = 0$) along the third axis, we come to the following expression for the derivative part of the effective Lagrangian

$$L_{\mathrm{der}}(B) = -\frac{e^2(\partial_i B)^2}{(8\pi)^2|eB|}\int_0^\infty\frac{d\omega}{\omega}\exp\left(-\frac{m^2\omega}{|eB|}\right)\times$$

$$\times\frac{d^3}{d\omega^3}(\omega\coth\omega). \tag{30}$$

Performing integration on the right-hand side of Eq. (3), we find the following representation in terms of special functions:

$$L_{\mathrm{der}}^{(3+1)}(B) = -\frac{e^2(\partial_i B)^2}{(8\pi)^2|eB|}\Bigg[\frac{11}{6}\left(\frac{m^2}{|eB|}\right)^3 + \left(\frac{m^2}{|eB|}\right)^2 - \frac{1}{3}\frac{m^2}{|eB|} - \left(\frac{m^2}{|eB|}\right)^3\psi\left(1 + \frac{m^2}{|eB|}\right) +$$

$$+ 24\,\zeta'\left(-2, 1 + \frac{m^2}{2|eB|}\right) - 24\frac{m^2}{|eB|}\times\zeta'\left(-1, 1 + \frac{m^2}{2|eB|}\right) + 6\left(\frac{m^2}{|eB|}\right)^2\times$$

$$\times\left[\ln\Gamma\left(1 + \frac{m^2}{2|eB|}\right) - \ln\sqrt{2\pi}\right]\Bigg], \tag{31}$$

where $\zeta'(s, q)$ is the derivative of the generalized zeta function over $s$, $\Gamma(z)$ is the Euler gamma function, $\psi(z) = d\ln\Gamma(z)/dz$.

As $m^2 \ll |eB|$, this expression allows the following asymptotic expansion:

$$L_{\mathrm{der}}(B) \approx \frac{e^2(\partial_i B)^2}{(8\pi)^2|eB|}\Bigg[24\,\zeta'(-2) + \frac{2m^2}{3|eB|} - \frac{m^4}{2|eB|^2} + \frac{m^6}{3|eB|^3} - \frac{m^8}{2|eB|^4}\sum_{k=0}^\infty\frac{k+1}{k+4}\times$$

$$\times\,\zeta(k+2)\left(-\frac{m^2}{2|eB|}\right)^k\Bigg], \tag{32}$$

where $\zeta'(-2) \approx -0.030$. As $m^2 \gg |eB|$, on the other hand, we obtain

$$L_{\mathrm{der}}(B) \approx -\frac{e^2(\partial_i B)^2}{(2\pi)^2m^2}\sum_{k=0}^\infty\frac{B_{2k+4}}{2k+1}\left(\frac{2|eB|}{m^2}\right)^{2k}. \tag{33}$$

In the case of the electric field along the first axis, on the other hand, we obtain the following expression for the derivative part of the effective action,

$$L_{\mathrm{der}}(E) = -\frac{i\,e^2(\partial_{||}E)^2}{(8\pi)^2|eE|}\int_0^\infty\frac{d\omega}{\omega}\exp\left(-i\frac{m^2}{|eE|}\omega\right)\times$$

$$\times \frac{d^3}{d\omega^3}(\omega \coth \omega). \qquad (34)$$

As expected in the case of an electric field background, this derivative correction to the effective Lagrangian contains both real and imaginary parts. A convenient representation of the latter can be obtained in the following way. First, in Eq. (34), we switch to a new variable, $z = i\omega$, so that the integration runs along the imaginary axis of $z$ from zero to $i\infty$. Then, we move the integration contour to the real axis of $z$, where the integrand has poles at $z = \pi n$ ($n$ = 1,2,...). As a result, the real and imaginary contributions get naturally separated. Indeed, the real part of $L_{\rm der}$ is given by the principal value of the integral along the $\mathrm{Re}(z)$ axis, while the imaginary part appears due to the integration along the infinite set of the vanishingly small semi-circles above the poles, $z = \pi n + \varepsilon \exp[i(\pi - \varphi)]$ (where $0 < \varphi < \pi$ and $\varepsilon \to 0$ at the end). Thus, for the imaginary part of the right-hand side in Eq. (34), we get

$$\mathrm{Im}\, L_{\rm der}(E) = \frac{e^2 (\partial_{||} E)^2}{2^6 \pi^4 |eE|} \sum_{n=1}^{\infty} \frac{1}{n^3} \exp\left(-\frac{\pi m^2 n}{|eE|}\right) \times$$

$$\times \left[6 + 6\frac{\pi m^2 n}{|eE|} + 3\left(\frac{\pi m^2 n}{|eE|}\right)^2 + \left(\frac{\pi m^2 n}{|eE|}\right)^3\right], \qquad (35)$$

which determines the correction to the probability of the particle-antiparticle pair creation (by definition, the probability density is $W = 2\,\mathrm{Im}\, L$) in an external electric field due to small inhomogeneities in space-time.

As is known, in the case of a constant electric field, the imaginary part of the effective Lagrangian is given by the Schwinger formula [4]

$$\mathrm{Im}\, L(E) = \frac{(eE)^2}{8\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{\pi m^2}{|eE|} n\right) =$$

$$= \frac{(eE)^2}{8\pi^3} \mathrm{Li}_2\left[\exp\left(-\frac{\pi m^2}{|eE|} n\right)\right], \qquad (36)$$

where $\mathrm{Li}_2(x)$ is the dilogarithmic function. As is easy to establish, both the Schwinger result for a constant field and the first correction due to derivatives are finite in the limit of the vanishing fermion mass.

Similar formulas were obtained also for scalar QED and for 2 + 1 scalar and spinor quantum electrodynamics [11]. It is clear that, in principle, an arbitrary finite order of the derivative expansion in our approach is computable. However, already the four-derivative term represents a formidable task for calculation by hands so that a computer assistance is needed.

In conclusion, we note that the developed technique can be applied to generating the derivative expansion around covariantly constant non-Abelian (see, for example [22, 23]) and gravitational fields (an analogue of Schwinger's formula in the case of covariantly constant Riemann curvature was obtained in [24]).

At the end, let us also make a few remarks about possible applications of the derivative expansion for the QED low-energy effective action. As in the case of Heisenberg — Euler action, the derivative corrections will affect, for example, the photon-photon scattering amplitude. For a vanishing background field, the later is discussed in [25]. Obviously, when the background field is nonzero, the corresponding amplitude and the energy dependence of the cross-section are going to change.

The derivative expansion might be useful in other problems, such as the generalization of the theory of magnetic catalysis of the chiral symmetry breaking in $\mathrm{QED}_4$ (see [8, 9, 26]) to the case of inhomogeneous magnetic fields.

I am grateful to the members of the Department of Physics of the Nagoya University, especially Prof. K. Yamawaki, for their hospitality during my stay there. The research has been supported in part by Deutscher Academischer Austauschdienst (DAAD) grant, by the National Science Foundation (USA) under grant No. PHY-9722059 and by the Grant-in-Aid of the Japan Society for the Promotion of Science No. 11695030.

1. *Weinberg S.* // Phys. Rev. — 1968. — **166.** — P. 1568: *Coleman S., Wess J., Zumino B.* // Ibid. — 1969. — **177.** — P. 2239.
2. *Heisenberg W., Euler H.* // Z. Phys. — 1936. — **98.** — P. 714 — 722.
3. *Weisskopf V.* // K. Dan. Vidensk. Selsk. Mat. -Fys. Medd. — 1936. — **14.** — P. 6 — 12.
4. *Schwinger J.* // Phys. Rev. — 1951. — **82**. — P. 664 — 679.
5. *Lee H.W., Pac P.Y, Shin H.K.* // Phys. Rev. D. — 1989. — **40.** — P. 4202 — 4205.
6. •µƒ·Ò ., ¤–¤ˇµ¢ . // . — 1970. — **11**, ¢Ò. 5. — . 1072 — 1077.
7. *Dunne G., Hall T.* // Phys. Lett. B. — 1998. — **419.** — P. 322; Phys. Rev. D. — 1999. — **60,** N 6. — P.065002.
8. *Gusynin V.P., Miransky V.A., Shovkovy I.A.* // Phys. Rev. D. — 1995. — **52.** — P. 4747 — 4751.
9. *Gusynin V.P., Miransky V.A., Shovkovy I.A.* // Nucl. Phys. B. — 1996. — **462.** — P. 249 — 290.
10. *Gusynin V.P., Shovkovy I.A.* // Can. J. Phys. — 1996. — **74.** — P. 282 — 289.
11. *Gusynin V.P., Shovkovy I.A.* // J. Math. Phys. — 1999. — **40.** — P. 5406 — 5439.
12. *Bern Z., Kosower D.A.* // Phys. Rev. Lett. — 1991. — **66.** — P. 1669 — 1672.
13. *Bern Z., Kosower D.A.* // Nucl. Phys. B. — 1992. — **379.** — P. 451.
14. *Strassler M.J.* // Ibid. — 1992. — **385**. — P. 145.
15. *McKeon D.G.C.* // Can. J. Phys. — 1993. — **71.** — P. 334; Ann. phys. — 1993. — **224.** — P.139.
16. *Schmidt M.G., Schubert C.* // Phys. Rev. D. — 1996. — **53.** — P. 2150.
17. *Körs B., Schmidt M. G.* // Eur. Phys. J. C. — 1999. — **6.** — P.175.

18. *Reuter M., Schmidt M.G., Schubert C.* // Ann. phys. – 1997. – **259**. – P. 313.
19. *Fradkin E.S.* // Nucl. Phys. – 1966. – **76**. – P. 588; *Fradkin E.S., Gitman D.M.* // Phys. Rev. D. – 1991. – **44.** – P. 3230.
20. *Fock V.A.* // Sov. Phys. – 1937. – **12.** – P. 404; *Pascual P., Tarrach R.* QCD Renormalization for the Practitioner: Lecture Notes in Physics. – **194**. – Berlin: Springer, 1984.
21. Éƒ¤· . ., ¡⁄ . . // . – 1971. – **33,** ¢. 5. – . 894 – 900.
22. *Gargett T.D., McArthur I.N.* // J. Math. Phys. – 1998. – **39.** – P. 4430.
23. *Banin A., Pletnev N.* // Phys. Rev. D. – 1999. – **60,** N 10. – P.105017.
24. *Avramidi I.G.* // J. Math. Phys. – 1996. – **37,** N 2. – P. 374.
25. *Dicus D.A., Kao C., Repko W.W.* // Phys. Rev. D. – 1998. – **57**. – P. 2443.
26. *Gusynin V.P.* // . – 2000. – **45,** ” 4 – 5. – . 603 – 609.

Received 04.05.00