AN ANALYTIC SOLUTION TO CERTAIN NONLINEAR EQUATIONS IN PLASMA PHYSICS

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We propose a new technology to analytically solve certain nonlinear equations arising in plasma physics. The results reveal that the present theory is convenient and efficient.

Below, we consider a problem of some importance in plasma physics. We are concerned with an electron beam injected into a plasma tube, where the magnetic field is cylindrical and increases towards the axis in inverse proportion to the radius. The beam is injected in parallel to the axis, but the magnetic field bends the path towards the axis. The governing equation for the path $u(x)$ of electrons is [1]

$$u'' + \varepsilon u^{-1} = 0, \tag{1}$$

with the initial conditions $u(0) = A$, $u'(0) = 0$.

In our study, the parameter ε does not need to be small, that is, $0 < \varepsilon < \infty$. In fact, the classical perturbation methods cannot be applied to such a problem even in the case where $\varepsilon \ll 1$. The problem can be solved by, for example, the variational iteration method [2], homotopy perturbation method [3, 4], linearization perturbation method [5], and others [6, 7]. Hereby we propose a new technology to solve the problem.

We rewrite Eq. (1) as

$$u'' + \varepsilon u^{-2} u = 0, \tag{2}$$

which can be approximated by the following equation:

$$u'' + \frac{\varepsilon u}{u_0^2} = 0, \tag{3}$$

where u_0 is an approximate solution of Eq. (1). In view of the physical understanding, we know that the system possesses a periodical solution. So we begin with

$$u_0 = A \cos \omega t, \tag{4}$$

where ω is the angular frequency of Eq. (1). Substituting u_0 into Eq. (3) results in

$$u'' + \frac{\varepsilon u}{A^2 \cos^2 \omega t} = 0, \tag{5}$$

or equivalently

$$u'' + \frac{2\varepsilon}{A^2} u + A^2 u'' \cos 2\omega t = 0. \tag{6}$$

Supposing that the solution of Eq. (6) can be decomposed as

$$u = u_0 + u_1, \tag{7}$$

where u_0 is a solution of the homogeneous equation

$$u_0'' + \frac{2\varepsilon}{A^2} u_0 = 0, \quad u_0(0) = A, \quad u_0'(0) = 0, \tag{8}$$

and u_1 is a solution of the following inhomogeneous equation:

$$u_1'' + \frac{2\varepsilon}{A^2} u_1 = -A^2 u_1'' \cos 2\omega t,$$

$$u_1(0) = 0, \quad u_1'(0) = 0. \tag{9}$$

The solution of Eq. (8) is

$$u_0 = A \cos \omega t, \tag{10}$$

where

$$\omega = \sqrt{2\varepsilon} / A. \tag{11}$$

So Eq. (9) can be rewritten as follows:

$$(1 + A^2 \cos 2\omega t)\, u_1'' + \omega^2 u_1 = A^2 \omega^2 \cos \omega t \cos 2\omega t,$$

$$u_1(0) = 0, \quad u_1'(0) = 0. \tag{12}$$

The above equation can be approximately solved as

$$u_1 = \frac{A^3}{2(1+4A^2)}(\cos\omega t - \cos 3\omega t). \quad (13)$$

So we obtain the following approximate solution:

$$u = u_0 + u_1 = A\cos\omega t + \frac{A^3}{2(1+4A^2)}(\cos\omega t - \cos 3\omega t). \quad (14)$$

The period of the system can be expressed in the form

$$T = \frac{\sqrt{2}\,\pi A}{\varepsilon^{1/2}} = \frac{4.442\,A}{\varepsilon^{1/2}}. \quad (15)$$

The exact period can be easily computed,

$$T_{ex} = 2\sqrt{2}\int_0^A \frac{du}{\sqrt{\int_u^A \varepsilon/s\,ds}} = \frac{2\sqrt{2}}{\sqrt{\varepsilon}}\int_0^A \frac{du}{\sqrt{\ln A - \ln u}}, \quad (16)$$

By the transformation $u = As$, Eq. (16) leads to

$$T_{ex} = \frac{2\sqrt{2}}{\sqrt{\varepsilon}} A\int_0^1 \frac{d\bar{u}}{\sqrt{\ln(1/s)}} = \frac{2\sqrt{2}}{\sqrt{\varepsilon}} A\sqrt{\pi} = \frac{5.01325\,A}{\varepsilon^{1/2}}. \quad (17)$$

Comparing with the exact solution reveals that the obtained result (15) is valid for all $\varepsilon > 0$. The 11.4% accuracy is remarkably good in view of its first-order approximation.

Hereby we propose a simple but efficient technique to search for an analytic solution of a nonlinear equation. The present method has the rapid convergence, and the first-order approximation obtained by the present technology is always of high accuracy. We can obtain readily a high-accuracy approximate solution by replacing $u_0 = A\cos\omega t + \frac{A^3}{2(1+4A^2)}(\cos\omega t - \cos 3\omega t)$ in Eq. (3). This new method will play an important role in determining many of the physical properties of nonlinear equations.

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