NON-UNITARY QUANTUM THEORY (MATHEMATICAL FOUNDATIONS). 1

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We present the mathematical apparatus of a new quantum scheme using a dual pair of reflexive topological vector spaces connected by any non-Hermitian form. Non-Fock (non-selfadjoint) representations of Heisenberg algebras and non-unitary representations, induced by them, of their automorphism groups underlie the theory, which gives a unique joint description of quantum objects and physical vacuum (ether).

1. Introduction

As is shown in [1], there is a strong theoretical support for the idea that, under extremal conditions (at supersmall distances or superhigh concentrations of energy), one has to take a non-standard dynamical system (called relativistic bi-Hamiltonian one described by the Heisenberg algebra $h_8^{(*)}$) into condideration. In the case, the Fock representation of the algebra $h_8^{(*)}$ and the unitary representation (generated by it) of its automorphism group $Sp^{(*)}(4, \mathbf{C})$ (a dynamical group of the system) was found to be incompatible to the condition of integrability of Hamiltonian flows p_μ and $\dot{p}_\mu$ in the representation space, i.e., in a Hilbert space H. Hence, it is necessary to use other representations together with a more general scheme (called non-unitary for brevity) of functional analysis based on the pair of topological vector spaces $(\dot{\mathbf{F}}, \mathbf{F})$ dual with respect to some non-Hermitian form $\langle .,. \rangle$. The theory of such representations is developed here.

Non-unitary quantum theory originated its beginning still in the 60st years when infinite-dimensional non-unitary representations of the rotation group and the Lorentz group called later as semispinor ones were considered in [2]. They result from the Dirac – Clifford operation applied to Grassmann spinors. Then a large part of these results was represented in difficultly accessible editions or in preprints.

The mathematical tools of a new quantum theory are developed here: non-Fock representations of those Heisenberg algebras being of interest from a physical standpoint are effeciently built, and their connection with extended Fock representations including some additional variables is established. The general purpose of this paper is to prove the theorem formulated here in a conditional form:

Non-unitary Quantum Theory = Unitary Quantum Theory + Hidden Parameters (*)

Reading this theorem from right to left allows one to say that the introduction of hidden (or additional) variables into the usual (unitary) quantum theory makes it non-unitary, i.e., as von Neumann [3] assumed, this results in a radical reorganization of the theory. A rather long way leads to the theorem. We have to go it. On this way, the notion of non-unitary quantum theory will be defined more procisely and those representations of the Heisenberg algebra and its automorphism group will be described that underlie the theory. It has to note that there is an own inner logic in the investigation and we follow it in our exposition. The paper is organized as follows. After a brief review in Section 2 of the Fock representation of Heisenberg algebras describing the standard oscillator whose spectrum is isomorphic to the standard model of natural number series, we move directly into Section 3 to the non-Fock representation and the non-standard oscillator having an infinite number of states with negative numbers of occupation and having no ground state. The main technical obstacle seems to be a cycling of creation and annihilation operators overcomed by means of a decycling operation. In Sections 4 – 6 (see part 2 of this paper), we compare Fock and non-Fock representations in detail. Finally, in sections 7 – 8 (see next paper), physical consequences are discussed.

2. Fock Representation of the Algebra h_2 and a Representation of the Algebra $sl(2, \mathbf{C})$ Related with it

1) We start our consideration of Heisenberg algebra representations connected in a certain way with Fock representations underlying the Heisenberg-Schrödinger unitary quantum theory from a brief mention about the latter.

The Heisenberg algebra h_{2n} is usually written in terms of $2n$ generators q_k, p_k $(k = 1,...,n)$ by commutation relations (h_{2n} is isomorphic to the nilpotent Lie algebra n_{3n}):

$$[q_j, p_k] = i\,\delta_{jk},\ [q_j, q_k] = [p_j, p_k] = 0. \tag{1}$$

It is convenient to go over to the Fock operators a_k^a where

$$a_k^1 = \frac{1}{\sqrt{2}}(q_k + ip_k),\quad a_k^2 = \frac{1}{\sqrt{2}}(q_k - ip_k) \tag{2}$$

obeyed the commutation relations

$$[a_k^a, a_{k'}^{a'}] = \delta_{kk'}\,\varepsilon^{aa'},\quad a, a' = 1, 2;\ k, k' = 1,\ldots, n \tag{3}$$

(here, $\delta_{kk'}$ is the Kronecker symbol, $\varepsilon^{aa'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the Levi – Civita symbol). The enveloping algebra $U[h_{2n}]$ is an infinite-dimensional Lie algebra which contains a finite-dimensional Lie subalgebra denoted by $l_{n(2n+3)+1}$ and called as a subalgebra of small oscillations, whose structure is described by the formula (the Levi – Mal'tzev decomposition)

$$l_{n(2n+3)+1} = (\sigma_{n(2n+1)} \oplus 1) +\!\!\!\!\supset h_{2n}, \tag{4}$$

where $\sigma_{n(2n+1)} \oplus 1 = h_{2n}^2$, and $\sigma_{n(2n+1)}$ are isomorphic to the Lie algebra $\mathrm{sp}(n, \mathbf{C})$ (h_{2n} is understood to be the complex algebra $h_{2n}(\mathbf{C})$). Generators of the algebra $\sigma_{n(2n+1)}$ are bilinear forms $a_k^a a_{k'}^{a'}$.

As the group $\mathrm{Sp}(n, \mathbf{C})$ is a group of internal automorphisms of the algebra $h_{2n}(\mathbf{C})$, we formally write

$$T(\upsilon)\, a_k^a\, T^{-1}(\upsilon) = \upsilon_{kk'}^{aa'}\, a_{k'}^{a'}, \tag{5}$$

where $\upsilon \in \mathrm{Sp}(n, \mathbf{C})$, and $T(\upsilon)$ is a 'spinor" (infinite-dimensional) representation of the group $\mathrm{Sp}(n, \mathbf{C})$. By definition, Lie brackets (3) are invariant under transformation (5).

The Fock representation of the algebra h_{2n} is built, as is known [4], in the space

$$\mathcal{F}_{\mathrm{F}} = \overline{U[a_k^2]}^{\,\tau}, \tag{6}$$

where $U[a_k^2]$ is the maximal commutative subalgebra in $U[h_{2n}]$ (a_k^2 are coordinates on the so-called Lagrangian plane), τ is a topology. In the Fock realization, we have $a_k^2 = z_k \in \mathbf{C}$ and $a_k^1 = \frac{\partial}{\partial z_k}$. In this case, the Hermitian inner product on $\mathcal{F}_{\mathrm{F}}$ is defined as

$$(f, g) = \int_{\mathbf{C}^n} \overline{f(z)}\, g(z)\, d\mu(z), \tag{7}$$

where $f, g \in \mathcal{F}_{\mathrm{F}}$ ($z = (z_1,\ldots, z_n) \in \mathbf{C}^n$), with the measure

$$d\mu(z) = \left(\frac{i}{2\pi}\right)^n \prod_{k=1}^{n} e^{-\bar{z}_k z_k}\, dz_k \wedge d\bar{z}_k \tag{8}$$

($\wedge$ is the external Cartan multiplication). In the Heisenberg-Schrödinger quantum theory, the Hilbert topology and the Hilbert space $\mathcal{F}_{\mathrm{F}} = \mathcal{H}$ are usually considered. An additional symmetry of the operators a_k^a or $a_k^1 = (a_k^2)^+$, where '+" is the conjugation for form (7), is connected with the inner product (7):

$$(a_k^2 f, g) = (f, a_k^1 g). \tag{9}$$

Therefore, the Fock representation deals with a certain real form of the algebra $h_{2n}(\mathbf{C})$ denoted as $h_{2n}(\mathbf{R})$ having $\mathrm{Sp}(n, \mathbf{R})$ as a group of automorphisms. As is known [4], the unitary two-valued representation $\upsilon \to T(\upsilon)$ of the group $\mathrm{Sp}(n, \mathbf{R}) \ni \upsilon$ is realized in $\mathcal{H}$. The Fock representation is known also to be unitary equivalent to the Schrödinger representation (1) (the Stone – von Neumann theorem about uniqueness of unitary representations of the Heisenberg group H_{2n}).

We are interested in other real forms of the algebra $h_{2n}(\mathbf{C})$ and the group $\mathrm{Sp}(n, \mathbf{C})$ (in particular, the compact group $\mathrm{USp}(n, \mathbf{C}) = \mathrm{Sp}(n)$) and their representation which have completely another nature: the operators $T(\upsilon)$ corresponding them, first, are unbounded on the representation space $\mathcal{F}$ (its topology will be determined further) and, secondly, are multiple-valued on the group.

The Fock representation of the algebra $h_{2n}(\mathbf{C})$ has a serious lack: in any sense, it is inconsistent (this inconsistency is eliminated further). To see this, it is enough to consider the case $n = 1$.

2) We recall that the square root operation (see [5]) applied to the Pauli bundle $E = (A_{3,1}, S_2, \bar{L})$ where the structural group $\bar{L}$ of a fiber S_2 is the group $\mathrm{SL}(2, \mathbf{C}) \approx \mathrm{Sp}(1, \mathbf{C})$ results us in the algebra $h_2(\mathbf{C})$.

As is known, the Fock representation of the algebra h_2 is built in the space

$$\mathcal{F}_{\mathrm{F}} = \bigoplus_{m=0}^{\infty} f_m \tag{10}$$

(here, $\oplus$ is understood to be an orthogonal sum), where $\{f_m\}$ is a canonical basis on which the operators a^α act under the law

$$0 \underset{a^1}{\leftarrow} f_0 \underset{a^1}{\overset{a^2}{\rightleftarrows}} f_1 \underset{a^1}{\overset{a^2}{\rightleftarrows}} \ldots\ f_m \underset{a^1}{\overset{a^2}{\rightleftarrows}} f_{m+1} \ldots \tag{11}$$

i.e., $a^2 f_m \sim f_{m+1}$, and $a^1 f_m \sim f_{m-1}$ (f_0 is called a ground state[1]: $a^1 f_0 = 0$). In the z-realization, we have

$$a^\alpha = \begin{pmatrix} d/dz \\ z \end{pmatrix}, \quad f_m = \frac{z^m}{\sqrt{m!}}, \quad m = 0, 1, 2, \ldots \tag{12}$$

Under the action of bilinear forms $a^\alpha a^\beta$, the space $\mathcal{F}_F$ breaks up into the orthogonal sum of two subspaces of even and odd (rather $z \to - z$) functions:

$$\mathcal{F}_F = \mathcal{F}^{(+)} \oplus \mathcal{F}^{(-)}.$$

Statement 1. Infinite-dimensional irreducible representations (corresponding to spins $-\frac{1}{4}$ and $-\frac{3}{4}$) of the Lie algebra $\mathrm{usp}(1, \mathbf{C}) \approx \mathrm{su}(2)$ are realized in $\mathcal{F}^{(+)}$ and $\mathcal{F}^{(-)}$, are given by the operators $\vec{L} = \frac{1}{4} a^\alpha \vec{\sigma}_\alpha^\beta a_\beta$, where $a_\beta = \varepsilon_{\beta\gamma} a^\gamma$ ($\varepsilon_{\beta\gamma} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\vec{\sigma}$ are the Pauli matrices), and are denoted $D^+(-\frac{1}{4})$ and $D^+(-\frac{3}{4})$ [6].

In the z-realization, we have ($L_\pm = L_1 \pm iL_2$)

$$L_3 = \frac{1}{2} z \frac{d}{dz} + \frac{1}{4}, \quad L_+ = \frac{1}{2} z^2, \quad L_- = -\frac{1}{2} \frac{d^2}{dz^2} \tag{13}$$

and the Casimir operator is $\vec{L}^2 \equiv -\frac{3}{16} = \lambda(\lambda + 1)$, whence it follows that $\lambda = -\frac{1}{4}, -\frac{3}{4}$.

In general case, $D^+(\lambda)$ is understood to be a representation of the algebra $\mathrm{su}(2)$ with the lowest Cartan vector (see [7] where such representations refer to semispinor ones) with spin λ. As $\mathrm{sl}(2, \mathbf{C}) = \mathrm{su}^c(2)$, the semispinor representation $D^+(\lambda)$ of the algebra $\mathrm{su}(2)$ extends obviously up to the representation $(\lambda, 0)^+$ of the algebra $\mathrm{sl}(2, \mathbf{C})$ [8].

Next, let us introduce the notations

$$\mathcal{F}^{(+)} = \mathcal{F}^{(+)}_{-\frac{1}{4}}, \quad \mathcal{F}^{(-)} = \mathcal{F}^{(-)}_{-\frac{3}{4}}.$$

By definition,

$$a^\alpha : \mathcal{F}^{(-)}_{-\frac{3}{4}} \leftarrow \mathcal{F}^{(+)}_{-\frac{1}{4}}, \tag{14}$$

so the operators a^α, changing a parity of space, lower a weight of representation (spin) by $\frac{1}{2}$: $-\frac{1}{4} \to -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4}$. On the other hand, in the Fock representation,

$$a^\alpha : \mathcal{F}^{(-)}_{-\frac{3}{4}} \to \mathcal{F}^{(+)}_{-\frac{1}{4}}, \tag{15}$$

i.e., the same operators raise a weight of representation by $\frac{1}{2}$: $-\frac{3}{4} \to -\frac{3}{4} + \frac{1}{2} = -\frac{1}{4}$. The phenomenon described by formulae (14), (15) refers us to a cycling of the operators a^α, and here is this cycle:

$$\mathcal{F}^{(-)}_{-\frac{3}{4}} \underset{a^\alpha}{\overset{a^\alpha}{\rightleftarrows}} \mathcal{F}^{(+)}_{-\frac{1}{4}}.$$

The given phenomenon is connected with a weight of representation (spin, as we actually consider representations of the rotation group $SO(3) \sim SU(2) \approx Sp(1)$) taking on values in the numerical field $\mathbf{Z}_2$ of the simple characteristic 2 [9]. Indeed, it is possible to present the pair of numbers $-\frac{1}{4}$ and $-\frac{3}{4}$ as $\lambda = -\frac{1}{4} - \frac{p}{2}$ where $p \in \mathbf{Z}_2 = \{0, 1\}$. Thus, we can write

$$\mathcal{F}_F = \bigoplus_{p \in Z_2} \mathcal{F}^{((-1)^p)}_{-\frac{1}{4} - \frac{p}{2}}. \tag{16}$$

It is an inconsistency of the Fock representation of the algebra h_2 from the point of view of the theory of spin that consists in that, in this representation, a spin takes its values from the field $\mathbf{Z}_2$ while it must take the values in the field of zero characteristic on physical reasons. This remark is extremely importance for the whole subsequent consideration.

The decycling operation to be formulated further is connected with a transition from the ring $\mathbf{Z}_2$ to the standard ring of zero characteristic $\mathbf{Z}$ which is connected with $\mathbf{Z}_2$ by the formula $\mathbf{Z}_2 = \mathbf{Z}/\mathrm{mod}\,2$. From the point of view of this formula, $\mathbf{Z}$ is a universal covering for $\mathbf{Z}_2$.

3. Decycling Operation. Non-Fock Representation of the Algebra h_4

1) At first, we describe a non-Fock representation of the algebra h_2 [9].

As is known, the Fock representation (denoted as $T_0(h_2)$) of the algebra h_2 plays an important role in the Heisenberg – Schrödinger quantum (unitary)

[1]The existence of the ground state is connected extremely with the condition for the Hermitian symmetry (9). Indeed, in the equation $a^2 a^1 f = mf$, there is the number $m = \frac{(a^1 f, a^1 f)}{(f, f)} \geq 0$ such that $m = 0$ if $a^1 f = 0$. So in this case, $m \in Z_+$.

theory. It is completely characterized by its ground state $|0\rangle$, called sometimes the mathematical vacuum, or the spectrum of the operator $\hat{N} = a_2 a_1 = a_1^+ a_1$, which is isomorphic to the standard model of natural number series $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$. The standard oscillator (it has a ground state) is described by this representation.

Another non-standard (or non-Fock) representation denoted by $T_1(h_2)$ is possible and underlies another quantum theory – the non-unitary one. A refusal from the additional condition for the Hermitian symmetry $a_2 = a_1^+$, and, hence, from the condition for a positive definite spectrum of the operator $\hat{N} = a_2 a_1$ results in the representation $T_1(h_2)$. Under these more general conditions, a spectrum of the operator $\hat{N}$, remaining equidistant, becomes unlimited in both sides, and, in general, it is complex: Spec $\hat{N} = \{\varepsilon + p\}_{p \in \mathbf{Z}}$. Weight vectors $|p\rangle$ of such (non-standard) an oscillator satisfy the equation $\hat{N}\,|p\rangle = (\varepsilon + p)\,|p\rangle$, and the spectrum of the operator $\hat{N}$ is isomorphic to a non-standard model of natural number series $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ($\mathbf{Z}$ satisfies the all arithmetics axioms, excepting the axiom of choice).

$\varepsilon \in \mathbf{C}/\mathbf{Z}$ associates with inequivalent representations such that a representation is irreducible for $\varepsilon \neq 0$ (for $\varepsilon = 0$ it is incompletely reducible, as there is an invariant subspace strained on vectors $|0\rangle, |1\rangle, |2\rangle, \ldots$). As is seen, h_2 has many non-standard representations, but all of them describe the same object – the non-standard oscillator.

The non-standard oscillator has an infinite set of states with negative numbers of occupation: $|-1\rangle, |-2\rangle, \ldots$ (which we call by states of physical vacuum or ether), and, hence, does not have a ground state (the first element is not present). $\mathbf{Z}$ is said to be ordered as a set, while $\mathbf{Z}_+$ is completely ordered. However, according to the Zermelo theorem, any set can be completely ordered. Our basic result – Theorem (*) about the existence of additional (hidden) parameters in the non-unitary theory – is predetermined by the Zermelo theorem (see Section 5).

2) We return to the Fock representation of the algebra h_2. Taking into account the physical nature of spin (see above), it is necessary to pass from the space $\mathcal{F}_{\mathrm{F}}$ to the space [10].

$$\mathbf{F}_z \equiv \bigoplus_{p \in \mathbf{Z}} \mathcal{F}^{((-1)^p)}_{-\frac{1}{4}+\frac{p}{2}} \qquad (17)$$

which is a kind of covering for the pair of spaces (16) (as $\mathbf{Z}$ is a covering for $\mathbf{Z}_2$). Here, $\oplus$ is understood to be a direct sum of spaces, as an opportunity of the usual sum is already exhausted (we would like to notice that, in the theory of representations of the group SO(2, 1) ~ SU (1, 1) ≈ Sp (1, $\mathbf{R}$), a weight of representation is not any spin, and, consequently, the expansion $\mathcal{F}_{\mathrm{F}} \subset \mathbf{F}_z$ is not obligatory in this case).

On the space $\mathbf{F}_z$ (17), the operators a^α act under the law

$$\ldots \underset{a}{\leftarrow} \mathcal{F}^{(+)}_{-\frac{5}{4}} \underset{a}{\leftarrow} \mathcal{F}^{(-)}_{-\frac{3}{4}} \underset{a}{\leftarrow} \mathcal{F}^{(+)}_{-\frac{1}{4}} \underset{a}{\leftarrow} \ldots\,,$$

i.e., they always only lower spin λ by $\frac{1}{2}$ $(\lambda \to \lambda - \frac{1}{2})$, so a^α: $\mathcal{F}^{((-1)^p)}_{-\frac{1}{4}+\frac{p}{2}} \to \mathcal{F}^{((-1)^{p-1})}_{-\frac{3}{4}+\frac{p}{2}}$. The operators raising spin by $\frac{1}{2}$ $(\lambda \to \lambda + \frac{1}{2})$ are denoted by b_α and act on $\mathbf{F}_z$ under the law:

$$\ldots \overset{b}{\to} \mathcal{F}^{(-)}_{-\frac{3}{4}} \overset{b}{\to} \mathcal{F}^{(+)}_{-\frac{1}{4}} \overset{b}{\to} \mathcal{F}^{(-)}_{\frac{1}{4}} \overset{b}{\to} \ldots\,,$$

i.e., b_α: $\mathcal{F}^{((-1)^{p-1})}_{-\frac{3}{4}+\frac{p}{2}} \leftarrow \mathcal{F}^{((-1)^p)}_{-\frac{1}{4}+\frac{p}{2}}$. The operators b_α were found in [11], and, in the z-realization, they are written down as

$$b_\alpha = {}^{(p)}b_\alpha = \frac{1}{2}\left(z, -\frac{d}{dz} + \frac{2p}{z}\right), \quad p \in \mathbf{Z}\,, \qquad (18)$$

With a^α, they are connected by the relation (it is checked directly)

$${}^{(p)}b_\alpha = z^{2p}\, a^{\beta}\, z^{-2p}\, \varepsilon_{\beta\alpha}. \qquad (19)$$

As is seen, it is the phenomenon of cycling that is connected with a degeneracy at $p = 0$: ${}^{(0)}b_\alpha = \frac{1}{2}\, a^{\beta}\, \varepsilon_{\beta\alpha}$.

Statement 2. On the space $\mathbf{F}_z$, the operators b_α satisfy the relations

$${}^{(p+1)}b_\alpha\, {}^{(p)}b_{\alpha'} - {}^{(p+1)}b_{\alpha'}\, {}^{(p)}b_\alpha = \varepsilon_{\alpha\alpha'},$$

and, hence, set some representation of the algebra h_2 on $\mathbf{F}_z$. This representation (as well as the representation given by the operator a^α) is incompletely reducible, as, in $\mathbf{F}_z$, there is a flag $0 \subset \ldots \subset \mathbf{F}_z^N \subset \mathbf{F}_z^{N+1} \subset \ldots \subset \mathbf{F}_z$ from h_2-invariant subspaces

$$\mathbf{F}_z^N = \bigoplus_{p=-\infty}^{N} \mathcal{F}^{((-1)^p)}_{-\frac{1}{4}+\frac{p}{2}}.$$

It is important to notice that the operators a and b do not commute among themselves. Further, these operators will be transformed into the operators φ and $\bar{\varphi}$ obeying the absolutely other commutation relations (20).

Now we note that space $\mathbf{F}_z$ contains subspaces of different parity having some inconveniences. Moreover, the z-realization possesses a defect such that the operators ${}^{(p)}b_\alpha$ belong not to the algebra $U[h_2]$, but the divisive body $K(h_2) = S^{-1}U[h_2]$ (introduced in [6]) builded over h_2 on the multiplicative $S = U[b_1]$. All this forces, making use of a connection between spaces of different parity $F_\lambda^{(-)} = zF_\lambda^{(+)}$, to go to spaces of a single, namely, positive parity. As

$$a^\alpha F_\lambda^{(+)} \subseteq F^{(-)}_{\lambda-\frac{1}{2}} = zF^{(+)}_{\lambda-\frac{1}{2}}$$

and

$$b_\alpha F_\lambda^{(-)} = b_\alpha zF_\lambda^{(+)} \subseteq F^{(+)}_{\lambda+\frac{1}{2}},$$

we have

$$\frac{1}{z}a^\alpha F_\lambda^{(+)} \subseteq F^{(+)}_{\lambda-\frac{1}{2}}, \quad b_\alpha zF_\lambda^{(+)} \subseteq F^{(+)}_{\lambda+\frac{1}{2}}.$$

Theorem 1. The operators

$$\varphi = \frac{1}{z}a^\alpha = \begin{pmatrix} z^{-1}d/dz \\ 1 \end{pmatrix},$$

$${}^{(p)}\bar{\varphi} = \frac{1}{2}{}^{(p)}b_\alpha z = \frac{1}{2}\left(z^2, -z\frac{d}{dz} + 2p + 1\right)$$

are densely defined on the space

$$\mathbf{F}_z^{(+)} = \bigoplus_{p\in\mathbf{Z}} F^{(+)}_{-\frac{1}{4}+\frac{p}{2}},$$

act on it under the law

$$\overset{{}^{(-1)}\bar{\varphi}}{\underset{\varphi}{\rightleftarrows}} F^{(+)}_{-\frac{3}{4}} \overset{{}^{(0)}\bar{\varphi}}{\underset{\varphi}{\rightleftarrows}} F^{(+)}_{-\frac{1}{4}} \overset{{}^{(1)}\bar{\varphi}}{\underset{\varphi}{\rightleftarrows}} F^{(+)}_{\frac{1}{4}} \overset{{}^{(2)}\bar{\varphi}}{\underset{\varphi}{\rightleftarrows}}$$

and obey the commutation relations

$$\varphi^\alpha\varphi^{\alpha'} - \varphi^{\alpha'}\varphi^\alpha = {}^{(p+1)}\bar{\varphi}_\alpha {}^{(p)}\bar{\varphi}_{\alpha'} - {}^{(p+1)}\bar{\varphi}_{\alpha'} {}^{(p)}\bar{\varphi}_\alpha = 0,$$

$$\varphi^\alpha {}^{(p+1)}\bar{\varphi}_{\alpha'} - {}^{(p)}\bar{\varphi}_{\alpha'}\varphi^\alpha = \delta^\alpha_{\alpha'}. \tag{20}$$

Let us introduce the operators a^a_α defining them by their restrictions on each of subspaces $F^{(+)}_{-\frac{1}{4}+\frac{p}{2}}$ $(p \in \mathbf{Z})$:

$$a^1_\alpha\Big|_{F^{(+)}_{-\frac{1}{4}+\frac{p}{2}}} = \varphi^\alpha, \quad a^2_\alpha\Big|_{F^{(+)}_{-\frac{1}{4}+\frac{p}{2}}} = {}^{(p)}\bar{\varphi}_\alpha. \tag{21}$$

(the language of Hopf algebras might be pertinent here, however, it is too cumbersome [12, 13]).

It follows from (20) that, on $\mathbf{F}_z^{(+)}$, the operators a^a_α satisfy the commutation relations

$$[a^a_\alpha, a^{a'}_{\alpha'}] = \delta_{\alpha\alpha'}\varepsilon^{aa'}, \tag{22}$$

and by that set some representation of the Heisenberg algebra h_4 on $\mathbf{F}_z^{(+)}$, which, as will be shown further, is not equivalent to the Fock representation of this algebra.

Let's assume $\frac{1}{2}z^2 = \zeta$. Then the operators φ^α, ${}^{(p)}\bar{\varphi}_\alpha$ can be written down in the form

$$\varphi^\alpha = \begin{pmatrix} d/d\zeta \\ 1 \end{pmatrix}, \quad {}^{(p)}\bar{\varphi}_\alpha = \left(\zeta, -\zeta\frac{d}{d\zeta} + p + \frac{1}{2}\right). \tag{23}$$

In this realization, the space of the representation is written down as

$$\mathbf{F}_\zeta = \bigoplus_{p\in\mathbf{Z}} F_{-\frac{1}{4}+\frac{p}{2}} \tag{24}$$

where $F_{-\frac{1}{4}+\frac{p}{2}}$ are classes of holomorphic functions of a complex variable ζ.

Points $-\frac{1}{4} + \frac{p}{2}$ of a complex plane of a spin variable λ have an unpleasant property that, between two points $-\frac{1}{4}$ and $-\frac{3}{4}$, the cycling still possible. To exclude this phenomenon forever, we replace $-\frac{1}{4}$ by a point of the general position $\lambda \neq -\frac{1}{4}, 0, \frac{1}{2}$, assuming a spin value in formulae (23), (24), to be equal to $\lambda + \frac{p}{2}$. We write down these formulae as

$$\varphi^\alpha = \begin{pmatrix} d/d\zeta \\ 1 \end{pmatrix}, \quad {}^{(p)}\bar{\varphi}_\alpha = \left(\zeta, -\zeta\frac{d}{d\zeta} + 2\lambda + p + 1\right),$$

$$\mathbf{F}_\zeta^{(\lambda)} = \bigoplus_{p\in\mathbf{Z}} F_{\lambda+\frac{p}{2}} \tag{25}$$

((25) transits into (23) at $\lambda = -\frac{1}{4}$). Thus, we have

$$\varphi^{\alpha}: \mathcal{F}_{\lambda+\frac{p}{2}} \to \mathcal{F}_{\lambda+\frac{p-1}{2}}; \quad {}^{(p)}\bar{\varphi}_{\alpha}: \mathcal{F}_{\lambda+\frac{p}{2}} \to \mathcal{F}_{\lambda+\frac{p+1}{2}}. \qquad (26)$$

4. Comparison of Representations

Statement 3. A representation of the algebra h_4 in the space $\mathbf{F}_{\zeta}$ is not equivalent to the Fock representation of this algebra, which is realized in the space $\mathcal{F}_{\mathrm{F}}$.

To see this, at first we give basic formulae of the Fock representation of the algebra h_4.

1) This representation is given by the operators

$$a^1_{\alpha} = \frac{\partial}{\partial z_{\alpha}} = \varphi^{\alpha}, \quad a^2_{\alpha} = z_{\alpha} = \bar{\varphi}_{\alpha}, \quad \alpha = 1, 2; \quad z_{\alpha} \in \mathbf{C} \qquad (27)$$

(see Section 2) acting on the Fock space $\mathcal{F}_{\mathrm{F}}$ formed by functions of two complex variables z_1, z_2. A canonical basis in $\mathcal{F}_{\mathrm{F}}$ consists of own functions of the operator $N = \sum_{\alpha=1,2} a^2_{\alpha} a^1_{\alpha}$, which are written down as

$$f^{m_2}_{m_1} = \frac{z_1^{m_1} z_2^{m_2}}{\sqrt{m_1!\, m_2!}}, \quad m_1, m_2 = 0, 1, 2, \ldots$$

In this basis, we have

$$\varphi^{(1)} f^{m_2}_{m_1} = \sqrt{m_1}\, f^{m_2}_{m_1-1}, \quad \bar{\varphi}_1 f^{m_2}_{m_1} = \sqrt{m_1+1}\, f^{m_2}_{m_1+1},$$

$$\varphi^{(2)} f^{m_2}_{m_1} = \sqrt{m_2}\, f^{m_2-1}_{m_1}, \quad \bar{\varphi}_2 f^{m_2}_{m_1} = \sqrt{m_2+1}\, f^{m_2+1}_{m_1}, \qquad (28)$$

whence it follows that the Fock representation describes a pair of standard oscillators having the same ground state f^0_0. The representation is denoted as $T_0(h_4)$.

An important characteristic of a representation of the algebra h_4 is which representation of the algebra su(2) giving by the operators $\vec{L} = \frac{1}{2} a^2 \vec{\sigma} a^1$ is realized in the representation space of the algebra h_4. In the Fock representation, the operators $\vec{L}$ are of the following form:

$$L_3 = \frac{1}{2}\left(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}\right), \quad L_+ = z_1 \frac{\partial}{\partial z_2}, \quad L_- = z_2 \frac{\partial}{\partial z_1}.$$

In $\mathcal{F}_{\mathrm{F}}$, they result in a representation of the algebra su(2) of the form

$$\bigoplus_{p \in \mathbf{Z}_+} D(p/2). \qquad (29)$$

Here, $D(p/2)$ is a finite-dimensional representation with spin $p/2$, which is realized in the subspace of homogeneous polynomials of a degree p in the form $\sum_{k=0}^{p} a_k z_1^k z_2^{p-k}$. The Casimir operator $\vec{L}^2 = L_0(L_0+1)$, where $L_0 = \frac{1}{2} a^2 a^1 = \frac{1}{2}\left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}\right)$, takes the values $\frac{1}{2} p \left(\frac{1}{2} p + 1\right)$ on such polynomials. Hence, on $\mathcal{F}_{\mathrm{F}}$, the representation of su(2) is completely reducible to its finite-dimensional representations.

A representation of the wider, Lorentz algebra sl(2, **C**) giving by the operators $\vec{L}, \vec{N}$, where $\vec{N} = \frac{i}{2}(a^2 \vec{\sigma} \varepsilon a^2 + a^1 \varepsilon \vec{\sigma} a^1)$, is characterized by the following values of the Casimir operators: $C = \vec{L}^2 - \vec{N}^2 = -\frac{3}{4}$, $C' = \vec{L}\vec{N} = 0$. On $\mathcal{F}_{\mathrm{F}}$, these operators set the representation $[\frac{1}{2}, 0] \oplus [0, \frac{1}{2}]$ [2], where $[\frac{1}{2}, 0], [0, \frac{1}{2}]$ are the well-known infinite-dimensional Majorana representations of the group SL(2, **C**) [14].

2) The representation of the algebra h_4 we considered (25) first differs from the Fock one by being given in the space of functions of one complex variable ζ instead of two. Moreover, it differs by the representation space $\mathbf{F}^{(\lambda)}_{\zeta}$, on which a representation of the algebra su(2) (given by the operators $\vec{L} = \frac{1}{2} a^2 \vec{\sigma} a^1$) is splitted into infinite-dimensional semispinor representations $D^+(\lambda)$ of the algebra su(2) according to the following formula [9]:

$$\bigoplus_{p \in \mathbf{Z}} D^+(\lambda + p/2). \qquad (30)$$

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[2] A weight of the representation $[\lambda_0, c]$ is connected with C and C' by the formulae $C = \lambda_0^2 + c^2 - 1$, $C' = i\lambda_0 c$, thus, the Pauli pair (λ, λ') is expressed by $[\lambda_0, c]$ by the formulae $\lambda = \frac{1}{2}(\lambda_0 + c - 1)$, $\lambda' = \frac{1}{2}(c - \lambda_0 - 1)$.

The representation $D^+(\lambda + p/2)$ is realized in the subspace $F_{\lambda+\frac{p}{2}}$ and is given by the operators $\vec{L}^{(\lambda+\frac{p}{2})}$ being a restriction of the operators $\vec{L}$ on the subspace $F_{\lambda+\frac{p}{2}}$: $\vec{L}^{(\lambda+\frac{p}{2})} = \vec{L}\Big|_{F_{\lambda+\frac{p}{2}}} = \frac{1}{2}\,{}^{(p)}\bar{\varphi}\,\vec{\sigma}\varphi$, where φ, ${}^{(p)}\bar{\varphi}$ are defined by formulae (25), (26), whence it follows that

$$L_3^{(\lambda+\frac{p}{2})} = \zeta\frac{d}{d\zeta} - \left(\lambda + \frac{p}{2}\right), \quad L_+^{(\lambda+\frac{p}{2})} = \zeta,$$

$$L_-^{(\lambda+\frac{p}{2})} = -\zeta\frac{d^2}{d\zeta^2} + 2\left(\lambda + \frac{p}{2}\right)\frac{d}{d\zeta} \tag{31}$$

being $\left(\vec{L}^{(\lambda+\frac{p}{2})}\right)^2 = \lambda(\lambda+1)$. As is seen, we deal here with a non-Lie realization (differential operators of the second order; compare with the Fock realization, formulae (27) – (29)). The Cartan – Weyl basis in $F_{\lambda+\frac{p}{2}}$ is formed by functions

$$f_m^{(\lambda+\frac{p}{2})}(\zeta) = (i\zeta)^m / \sqrt{m!\,\Gamma(m - 2\lambda - p)},$$

$$m = \mathbf{Z}_+, \quad p \in \mathbf{Z}, \tag{32}$$

where Γ is the Euler function normalized by the condition

$$\langle f_m^{(\bar{\lambda})}, f_{m'}^{(\lambda)}\rangle_\lambda = (-1)^m\,\delta_{mm'},$$

where

$$\langle f, g\rangle_\lambda = \int \overline{f(\zeta)}\,Ig(\zeta)\,d\mu_\lambda(\zeta), \tag{33}$$

$Ig(\zeta) = g(-\zeta)$, and

$$d\mu(\zeta) = \frac{i}{\pi}\frac{1}{|\zeta|^{2\lambda+1}}K_{2\lambda+1}(2|\zeta|)\,d\zeta \wedge d\bar{\zeta} \tag{34}$$

(K_a is the Macdonald function, this measure is a generalization of the Gauss (8) measure on the case of any spin λ) is an su(2)-invariant sesquilinear form on $F_{\lambda+\frac{p}{2}}$ determined in [13, 15]: $\langle f^{(\bar{\lambda})}, \vec{L}^{(\lambda)}g^{(\lambda)}\rangle_\lambda = \langle \vec{L}^{(\bar{\lambda})}f^{(\bar{\lambda})}, g^{(\lambda)}\rangle_\lambda$. Thus, the operators $\vec{L}^{(\bar{\lambda})}$, answering a complex conjugate spin $\bar{\lambda}$, set a conjugate representation $D^+(\bar{\lambda})$ in the dual space $F_{\bar{\lambda}}$. It is necessary to notice that form (33) is not invariant under transformations $\mathrm{su}^c(2) = \mathrm{su}(2) + i\,\mathrm{su}(2)$.

On the whole space $\mathbf{F}_\zeta^{(\lambda)}$, the sesquilinear form is given by the sum

$$\langle f, g\rangle = \sum_{p=-\infty}^{\infty}\langle f^{(\bar{\lambda}+\frac{p}{2})}, g^{(\lambda+\frac{p}{2})}\rangle_{\lambda+\frac{p}{2}}, \tag{35}$$

where $g^{(\lambda+\frac{p}{2})}$ is a projection of g onto the space $F_{\lambda+\frac{p}{2}}$. Thus, it is obvious that $\langle f, \vec{L}g\rangle = \langle \vec{L}f, g\rangle$. Moreover, form (35) is invariant under the wider algebra sp(2, **R**), generators of which are $I_{\mu\nu} = (\vec{L}, \vec{N})$, $\Gamma_\mu = (\vec{\Gamma}, \Gamma_0)$ where $\vec{N} = \frac{i}{4}(a^1\varepsilon\vec{\sigma}a^1 + a^2\vec{\sigma}\varepsilon a^2)$, $\vec{\Gamma} = \frac{1}{4}(a^1\varepsilon\vec{\sigma}a^1 - a^2\vec{\sigma}\varepsilon a^2)$, $\Gamma_0 = L_0 + \frac{1}{2}$, and $L_0 = \frac{1}{2}a^2a^1 = \frac{1}{2}N$ such that $\vec{L}^2 = L_0(L_0 + 1)$. In the given representation, the Casimir operators have the same values $C = \vec{L}^2 - \vec{N}^2 = -\frac{3}{4}$, $C' = \vec{L}\vec{N} = 0$, $\Gamma_\mu^2 = \vec{\Gamma}^2 - \Gamma_0^2 = \frac{1}{2}$, as in the Fock representation, though the representations, as we see, are different. The point is that values of the Casimir operators do not define a representation of the algebra sl(2, **R**) completely. The spectrum of the operator L_0 is also important. It consists of points $\mathrm{Spec}\,L_0 = \{\lambda + \frac{p}{2}\}_{p\in\mathbf{Z}}$ on the space $\mathbf{F}_\zeta^{(\lambda)}$, while, on the space F_F, it is formed by points $\mathrm{Spec}\,L_0 = \{\frac{p}{2}\}_{p\in\mathbf{Z}_+}$. The representation of the algebra sl(2, **C**) on $\mathbf{F}_\zeta^{(\lambda)}$ is denoted as $[\frac{1}{2}, 0]^+ \oplus [0, \frac{1}{2}]^+$, where $[\frac{1}{2}, 0]^+$ is an infinite-dimensional 'tail" of the Majorana representation $[\frac{1}{2}, 0]$ unequivalent to the Majorana representation. By the restriction $\mathrm{sl}(2, \mathbf{C}) \supset \mathrm{su}(2)$, the given representation decays into irreducible semispinor representations of su(2) under formula (30).

At last, we mention one more remark. Let us denote $f_m^\mu = f_m^{(\lambda+\frac{p}{2})}$, putting $\mu = 2\lambda + p - m$. In this basis, the operators a_α^a are given by the formulae [9]:

$$a_1^1 f_m^\mu = \sqrt{m}\,f_{m-1}^\mu, \quad a_1^2 f_m^\mu = \sqrt{m+1}\,f_{m+1}^\mu,$$

$$a_2^1 f_m^\mu = \sqrt{\mu}\,f_m^{\mu-1}, \quad a_2^2 f_m^\mu = \sqrt{\mu+1}\,f_m^{\mu+1}, \tag{36}$$

whence it follows that the operators (a_1^1, a_1^2) are related to the standard oscillator (there is a ground state, as $m = 0, 1, 2, \ldots$), while (a_2^1, a_2^2) apply to the non-standard oscillator (a ground state is not present, as $p = 0, \pm 1, \pm 2, \ldots$). Such a representation of h_4 is denoted as $T_1(h_4)$ and called non-Fock one. A representation, in which both oscillators are non-standard, is denoted by $T_2(h_4)$.

Now it is possible to say definitely that the representations $T_0(h_4)$ and $T_1(h_4)$ are not equivalent (Statement 3).

In general, the algebra h_{2n} has $n+1$ representations $T_k(h_{2n})$ (unequivalent among themselves, where $k\,(0 \le k \le n)$ is the number of non-standard oscillators.

We give yet a precise definition of decycling for the Fock representation $T_0(h_{2n})$. The maximal decycling is understood to be a transit from the Fock representation $T_0(h_{2n})$ of the algebra h_{2n} to the non-Fock representation $T_n(h_{4n})$ of the algebra h_{4n} in a twice greater dimension.

3) At the end of this section, we stop on the description of semispinor representations of the algebra su(2), its complex expansions $\mathrm{su}^c(2)$, and also the group $\mathrm{SU}^c(2)$ as a whole. As we have already noticed, in the direct sum (25), each of subspace F_λ is invariant under the envelope algebra $U[\vec{L}^{(\lambda)}]$ and, hence, $\mathrm{su}^c(2)$-invariantly (in F_λ generators of $\mathrm{su}^c(2)$ are the operators $(\vec{L}^{(\lambda)}, i\vec{L}^{(\lambda)})$, and as the semispinor representation $(\lambda, 0)^+ = [\lambda, \lambda+1]^+$ of the algebra $\mathrm{su}^c(2)$, called analytical one, is realized in F_λ, so we have the representation $\bigoplus_{p\in \mathbf{Z}} [\lambda + \frac{p}{2}, \lambda + \frac{p}{2} + 1]^+$ in $\mathbf{F}\zeta$). A complexification of the algebra su(2) is important in the connection with the problem of relativization of spin[3]. Semispinor representations $(\lambda, 0)^+$ of the algebra $\mathrm{su}^c(2)$ are representations spontaneously breaking the symmetry of the group $\mathrm{SU}^c(2)$ as a whole. Indeed, we shall now see that the Lie algebra $(\vec{L}^{(\lambda)}, i\vec{L}^{(\lambda)})$ generates no representations of the group $\mathrm{SU}^c(2)$, but only ones of its open subgroups connected with the Gauss decomposition $N_+ H N_-$ of this group.

As is known, any regular element $\upsilon = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $(\delta \ne 0)$ of the group SL(2, **C**) allows the decomposition $\upsilon = n_+ h n_-$, where

$$n_+ = \begin{pmatrix} 1 & \beta/\delta \\ 0 & 1 \end{pmatrix} = \exp\left(\frac{\beta}{\delta}\sigma_+\right),$$

$$h = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix} = \exp(-\sigma_3 \ln \delta),$$

$$n_- = \begin{pmatrix} 1 & 0 \\ \gamma/\delta & 1 \end{pmatrix} = \exp\left(\frac{\gamma}{\delta}\sigma_-\right),$$

and $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If to remove singular elements $\begin{pmatrix} \alpha & \beta \\ -\beta^{-1} & 0 \end{pmatrix} \in \Delta'$ from SL(2, **C**), SL(2, **C**) will break up on two open Borel subgroups $B_+ = N_+ H \ni \begin{pmatrix} \delta^{-1} & \beta \\ 0 & \delta \end{pmatrix}$ and $B_- = HN_- \ni \begin{pmatrix} \delta^{-1} & 0 \\ \gamma & \delta \end{pmatrix}$ which are crossed through the subgroup H of diagonal matrices forming the Gaussian area $B_+ B_-$. The matrices $\sigma_+, \sigma_-, \frac{1}{2}\sigma_3$ are the generators of the subgroups N_+, N_-, H. The representations in question are given by mapping (homomorphism) T_λ: $\frac{1}{2}\vec{\sigma} \to {}^{(p)}\bar{\varphi}\,\frac{1}{2}\vec{\sigma}\varphi = \vec{L}^{(\lambda + \frac{p}{2})}$, where φ, ${}^{(p)}\bar{\varphi}$ are defined by (25), (26), and $\vec{L}^{(\lambda + \frac{p}{2})}$ are given by (31). The operators $\vec{L}^{(\lambda)}$ set the semispinor representation $D^+(\lambda)$ of the algebra su(2) (or the representation $(\lambda, 0)^+$ of the algebra $\mathrm{su}^c(2)$) in the space F_λ.

Consider first the subgroup B_+. As T_λ is a homomorphism, we have $T_\lambda(h_+) = T_\lambda(n_+)\,T_\lambda(h)$, where [9]

$$T_\lambda(h) = e^{-T_\lambda(\sigma_3)\ln\delta} = e^{-2L_3^{(\lambda)}\ln\delta},$$

$$T_\lambda(n_+) = e^{T_\lambda(\sigma_+)\beta/\delta} = e^{(\beta/\delta)L_+^{(\lambda)}} = e^{(\beta/\delta)\,\zeta}. \tag{37}$$

As $L_3^{(\lambda)}\zeta^n = (n-\lambda)\zeta^n$ (see (31)),

$$T_\lambda(b_+)\,f(\zeta) = \delta^{2\lambda} e^{\beta\zeta/\delta} f\left(\frac{\zeta}{\delta^2}\right) \tag{38}$$

where $f(\zeta) \in F_\lambda$. It follows from (38) that a class of holomorphic functions of a complex variable ζ of order $\rho \le 1$ and type $0 \le \tau < \infty$ is invariant under the action of operators $T_\lambda(b_+)$. This class allows a topologization transforming it into the space of generalized functions of the exponential type Φ' (see a definition of the space Φ' in [16]). By definition, we have $\Phi' = \overline{U[L_+^{(\lambda)}]\,f_0^{(\lambda)}}^{\tau_{\Phi'}}$, where $f_0^{(\lambda)} = 1$ is the lowest Cartan – Weyl vector of the representation $D^+(\lambda)$

[3] The algebra $\mathrm{su}^c(2) \approx \mathrm{sl}(2, \mathbf{C})$ should not be mixed up with another algebra sl(2, **C**) represented in F_λ by the operators $(\vec{L}, \vec{N})$, see above.

(a cyclic vector for $U(L_+^{(\lambda)}])$, and $\tau_{\Phi'}$ is a topology of the space Φ' determined in [16].

The space Φ' is determined so that an B_+-orbit of any vector $f \in \Phi'$ (including any vector of the Cartan – Weyl basis $f_n^{(\lambda)}$) wholly lies in Φ', as $e^{\tau L_+^{(\lambda)}} f(\zeta) = e^{\tau\zeta} f(\zeta) \in \Phi'$ $(|\tau| < \infty)$. In Φ', a representation of the (solvable) group B_+ is incompletely reducible, as there exists a compositional series $\Phi' = \Phi'_0 \supset \Phi'_1 \supset \dots \supset \Phi'_N \supset \dots \supset 0$ in Φ', where $\Phi'_N = \overline{\bigoplus_{n=N}^{\infty} f_n^{(\lambda)}}^{\tau_{\Phi'}}$ are B_+-invariant subspaces. The operators $T_\lambda(b_+)$ set (in F_λ) the exact representation of the universal covering $\overline{B}_+ = N_+\overline{H}$, where $\overline{H}$ is an universal covering for H.

We now consider the subgroup B_-. As

$$e^{\tau L_-^{(\lambda)}} \zeta^n = n!\left(-\frac{\tau}{2}\right)^n L_n^{(-2\lambda-1)}\left(\frac{\zeta}{\tau}\right) \tag{39}$$

($L_n^{(\lambda)}$ is Laguerre polynomials [17]), in the linear system (l.s.) $\{f_n^{(\lambda)}\}$, there is a flag from finite-dimensional subspaces: $0 \subset \Phi^0 \subset \Phi^1 \subset \dots \subset \Phi^N \subset \subset \dots \subset U[L_+^{(\lambda)}]\,\Phi^0$, where $\Phi^N = \bigoplus_{n=0}^{N} f_n^{(\lambda)}$ are B_--invariant subspaces. Hence, an incompletely reducible representation $B_- \to T_\lambda(b_-)$ of the group B_- is realized in l.s. $\{f_n^{(\lambda)}\}$. It is not difficult to show that l.s. $\{f_n^{(\lambda)}\}$ allows a B_--invariant closure up to the space Φ of trial functions of the exponential type containing holomorphic functions of a complex variable ζ of order $\rho < 1$ and type $0 \le \tau < \infty$ (see a definition of space Φ in [16]). Indeed, from the definition of Φ' as a space of linear continuous functionals on the space Φ

$$f(\varphi) = \langle f, \varphi\rangle_\lambda, \tag{40}$$

where $\langle .,. \rangle_\lambda$ is defined by formula (33), and from properties of $\overrightarrow{L}^{(\lambda)}$-invariance of this functional, it follows that $\langle \Phi', e^{\tau L_-^{(\lambda)}} \Phi\rangle = \langle e^{\bar{\tau} L_+^{(\lambda)}} \Phi', \Phi\rangle$. As we have $e^{\bar{\tau} L_+^{(\lambda)}} \Phi' \subset \Phi'$ for any finite τ, so $e^{\tau L_-^{(\lambda)}} \Phi \subset \Phi$.

It is interesting to observe that so-called coherent states $f_\sigma^{(\lambda)}(\zeta) = {}_0F_1(-2\lambda; \sigma\zeta)$ satisfying the equation $L_-^{(\lambda)} f_\sigma^{(\lambda)}(\zeta) = \sigma f_\sigma^{(\lambda)}(\zeta)$, $\sigma \in \mathbf{C}$ belong to the space Φ. On Φ, the operators $T_\lambda(b_-)$ set the exact representation of the universal covering $\overline{B}_- = \overline{H}N_-$.

Further, it is clear that the operators $T_\lambda(N_+)$ are not determined on the space Φ: each vector $f \in \Phi$ is removed at once by the operator $T_\lambda(n_+) = e^{\zeta\beta/\delta}$ $(\beta \ne 0)$ from Φ. In particular, (39) and (38) imply the formula

$$T_\lambda(\upsilon)\,\zeta^n = n!\,\delta^{2\lambda}\left(-\frac{\gamma}{2\delta}\right)^n L_n^{(-2\lambda-1)}\left(\frac{\zeta}{\gamma\delta}\right) e^{\beta\zeta/\delta}.$$

On the other hand, each vector from Φ' not belonging to the subset $\Phi \subset \Phi'$ may be removed by the operator $T_\lambda(n_-)$ from Φ'. Indeed, it follows from the formula

$$T_\lambda(n_-)\, e^{\zeta\tau} = (\gamma\tau + 1)^{2\lambda} e^{\zeta\frac{\tau}{\tau\gamma+1}}$$

or from the more general formula $T_\lambda(n_-)\,\Phi\, e^{\zeta\tau} = \Phi T_\lambda(n_-)\, e^{\zeta\tau}$ that, for $\gamma = -\frac{1}{\tau}$, the vector $T_\lambda(n_-)\, e^{\zeta\tau} \notin \Phi'$. Thus, on Φ', there is no representation (as a linear one) of the group N_- in the usual sence.

Actually, a non-Neumann representation (see its definition in [18]) of the group $\widetilde{N}_-$ of one-dimensional chains (and also the 1-chain group $\widetilde{\mathrm{SL}}(2, \mathbf{C})$, see its definition in [18]) which certainly is not any manifold (group $N_\mp$ as well as $\mathrm{SL}(2, \mathbf{C})$ is simply connected) is realized in Φ'. Thus, remaining in the framework of the topological vector space Φ', it will be impossible to close the area B_+B_- to obtain the group $\mathrm{SL}(2, \mathbf{C})$ and its representation in Φ'. It follows from the formula

$$T_\lambda(\upsilon)\, e^{\zeta\tau} = (\gamma\tau + \delta)^{2\lambda} e^{\zeta\frac{\alpha\tau+\beta}{\gamma\tau+\delta}},$$

$$\upsilon = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C}) \tag{41}$$

that, in Φ', we have representations of the 1-chain group $\widetilde{\mathrm{SL}}(2, \mathbf{C})$ not satisfying the von Neumann axiom about the existence of a common (dense in Φ') domain of definition for all the operators $T_\lambda(\upsilon)$. It follows from (41) that such a domain is trivial, i.e., $\bigcap_{\upsilon \in \mathrm{SL}(2,\mathbf{C})} D_{T_\lambda(\upsilon)} = 0$; therefore, the corresponding representations were called by non-Neumann ones [18]. Non-Neumann representations of topological groups are a subject of the special discussion which we shall not touch upon here.

So, the non-Fock representation of the algebra h_4 built in the dual pair of topological vector spaces (Φ', Φ), generates such a representation of its automorphisms group $\mathrm{Sp}(2, \mathbf{R})$ which breaks this group symmetry: the nature of the representation is such that the complete symmetry is spontaneously lowered up to the symmetry of open subgroups B_+ and B_- connected with the Gauss decomposition. The group $B_+ = N_+ H$ can be represented only in the space Φ', and $B_- = HN_-$ only in the space Φ. (It is interesting to note that the existence of open subgroups in Lie's group was always a mystery, that is disclosed in semispinor theory only, describing a new physical entity). The operators $L_\pm = L_1 \pm iL_2$ and $\overrightarrow{\Gamma} \pm i\overrightarrow{N}$ are the generators of the subgroups $N_\pm$, and L_3 and Γ_0 are the generators of H. Thus, semispinors are objects with a restricted symmetry. And there are two kinds of semispinors: semispinors having the symmetry of the group B_+ (associated with the Gauss decomposition $N_+ HN_-$ and the pair of spaces (Φ', Φ)) and semispinors having the symmetry of the group B_- (associated with another Gauss decomposition $N_- HN_+$ and another pair of spaces (Φ, Φ')). Both the pairs of spaces and both the Gauss decompositions are connected by the time reflection operator T for which formulae are read from the right on the left. So, only at the level of semispinors, we have an opportunity to break both T-symmetry and CP-symmetry which are impossible to break at a level of spinors and spinor fields (see [9]): the theory with fixed pair of spaces is asymmetric.

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