TWO-FLUID COSMOLOGICAL MODELS IN THE EINSTEIN – CARTAN THEORY WITH TWO SOURCES OF TORSION

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A variant of the Einstein – Cartan theory, in which a perfect fluid and a non-minimal coupling scalar field are the sources of torsion, is considered. The exact solutions for closed cosmological models with allowance for the stiff fluid have been obtained. It is shown that non-singular models are possible in some cases. The influence of material sources on the character of evolution of the models is discussed.

Introduction

It is well known that the problems of general relativity (GR) and the standard cosmological scenario stimulated the development of other relativistic theories of gravity. Some progress has been made in the actual variant of the Poincare gauge theory of gravity, the Einstein – Cartan theory (ECT), in eliminating certain difficulties of GR, and in constructing viable cosmological models (see, for example, [1 – 4] and references therein).

In the present paper, we investigate the variant of ECT, which simultaneously takes into consideration two sources of torsion: a perfect fluid and a non-minimally coupled scalar field. These sources rather ofter have been considered as models of matter distributions, for example, in GR. Supplementary interest to a non-minimally coupled scalar field in the relativistic theories of gravity is connected with its role in inflationary cosmology and its presence in GUT models and Kaluza – Klein theories.

In the framework of two-torsion ECT, we consider homogeneous isotropic closed cosmological models with a non-minimally coupled scalar field with a nonlinear potential and two perfect fluids, one of which is a stiff fluid and the other is the source of torsion.

1. Basic Equations

The Lagrangian L of the model is chosen as the sum of Lagrangians: gravitational – L_g, scalar field – L_s, and perfect fluids – $L_{\mathrm{fl}(1)}$ and $L_{\mathrm{fl}(2)}$:

$$L_g = - R(\Gamma)/2\kappa, \tag{1}$$

$$L_s = \frac{1}{4\pi}\left\{\frac{\alpha_s}{2}\left[\Phi_{,k}\Phi^{,k} + \xi R(\Gamma)\Phi^2\right] - V(\Phi)\right\}, \tag{2}$$

$$L_{\mathrm{fl}(1)} = -\rho\left(c^2 + \Pi(\rho, e)\right) + k\overset{\Gamma}{\nabla}_i(\rho u^i) +$$

$$+k_1(u_i u^i - 1) + k_2 u^i\partial_i X + k_3 u^i \partial_i e. \tag{3}$$

Here, $R(\Gamma)$ is the curvature scalar obtained from the full connection $\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} + S^k_{ij\cdot} + S^{\;k}_{\cdot\,ij} + S^{\;k}_{\cdot\,ji}$; $\left\{ {k \atop ij} \right\}$ are Christoffel symbols of the second kind; $S^k_{ij\cdot} = \Gamma^k_{[ij]}$ is the torsion tensor; $\kappa = 8\pi G/c^4$ is the Einstein's constant; $\alpha_s = +1$ conforms to the material scalar field; $\alpha_s = -1$ corresponds to the 'gravitational" scalar field [3 – 6]; ξ is a coupling constant; $V(\Phi)$ is the potential of the scalar field; ρ is the perfect fluid mass density; $\Pi(\rho, e)$ is its internal energy; k, k_1, k_2, k_3 are the Lagrange multipliers; X is the Lagrangian coordinates of the matter particles; e is the entropy per volume [7]; u^i is the four-velocity; $\overset{\Gamma}{\nabla}_i$ is the covariant derivative of the Riemann – Cartan space. The Lagrangian $L_{\mathrm{fl}(2)}$ for the stiff fluid is not indicated since there is no torsion vector for it in the derivative of the term, which regulates the conservation of the number of particles.

In the paper, the metric g_{ik} has signature (-, -, -, +), the Riemann and Ricci tensors are defined as $R^{m}_{ijk\cdot} = \Gamma^{m}_{jk,i} - \Gamma^{m}_{ik,j} + \Gamma^{m}_{ip}\Gamma^{p}_{jk} - \Gamma^{m}_{jp}\Gamma^{p}_{ik}$, $R_{jk} = R^{i}_{ijk\cdot}$. It follows from (3) that the torsion can interact with a perfect fluid only through its trace: $S_i = S^{k}_{ik\cdot}$ (the vector of torsion). An analogous result has been derived [5] for a scalar field. Hence, the curvature scalar can be presented in the form:

$$R(\Gamma) = R(\{\}) + 4S^{k}_{;k} - \frac{8}{3}S_k S^k, \tag{4}$$

were $R(\{\})$ is the Riemannian part of the curvature built from Christoffel symbol; semicolon denote the covariant derivative of the Riemannian space.

One can note that Lagrangian (2) in the torsionless case at $\xi = 1/6$ and $V(\Phi) = 0$ conforms to the conformally invariant scalar field. As shown in [5], when $\alpha_s = -1$, $\xi = -1/6$, $V(\Phi) = -\frac{1}{2}\mu^2\Phi^2$, the scalar field corresponding to Lagrangian (2) is an axion field in GR. From the viewpoint of QCD, the interest to the axion field is based on the fact, that it leads to a compensation of strong CP violation effects; from the viewpoint of cosmology, it is a cold dark matter candidate (see, for example, [5, 8] and references therein).

Varying the action with the Lagrangian $L = L_g + L_s + L_{\mathrm{fl}(1)} + L_{\mathrm{fl}(2)}$ in g_{ij}, S_k, Φ, ρ, k, k_i, X, e, u^i, we obtain the following set of equations for the gravitational fields and matter:

$$G_{ij}(\{\}) = \kappa(T^{\mathrm{fl}(1)}_{ij} + T^{\mathrm{fl}(2)}_{ij} + T^{s}_{ij}) + \Lambda_{ij}, \tag{5}$$

$$S^k = \frac{3}{2}\Psi(2\pi\alpha_s\Theta u^k + \xi\Phi\Phi^{,k}), \tag{6}$$

$$\Box\Phi - \xi\Phi R(\Gamma) + \alpha_s V' = 0, \tag{7}$$

$$\varepsilon_{\mathrm{fl}(1)} + P_{\mathrm{fl}(1)} + \rho u^i(\partial_i + 2S_i)k = 0, \tag{8}$$

$$\overset{\Gamma}{\nabla}_i(\rho u^i) = 0, \tag{9}$$

$$u_i u^i = 1, \tag{10}$$

$$u^i\partial_i X = 0, \tag{11}$$

$$u^i\partial_i e = 0, \tag{12}$$

$$(k_2 u^i)_{;i} = 0, \tag{13}$$

$$\partial\varepsilon_{\mathrm{fl}(1)}/\partial e + (k_3 u^i)_{;i} = 0, \tag{14}$$

$$-\rho\partial_i k - 2k\rho S_i + 2k_1 u_i + k_2\partial_i X + k_3\partial_i e = 0, \tag{15}$$

where

$$T^{\mathrm{fl}(1)}_{ij} = (\varepsilon_{\mathrm{fl}(1)} + P_{\mathrm{fl}(1)})u_i u_j - P_{\mathrm{fl}(1)}g_{ij}, \tag{16}$$

$$T^{\mathrm{fl}(2)}_{ij} = (\varepsilon_{\mathrm{fl}(2)} + P_{\mathrm{fl}(2)})u_i u_j - P_{\mathrm{fl}(2)}g_{ij}, \tag{17}$$

$$T^{s}_{ij} = \frac{\alpha_s}{4\pi}\{\Phi_{,i}\Phi_{,j} - \frac{1}{2}[\Phi_{,m}\Phi^{,m} + \xi R(\{\})\Phi^2 -$$

$$- 2\alpha_s V(\Phi)]g_{ij} + \xi[-2S_i\nabla_j - 2S_j\nabla_i + 2g_{ij}S^n\nabla_n -$$

$$- \nabla_i\nabla_j + g_{ij}\Box + R_{ij}(\{\}) - \Lambda_{ij}]\Phi^2\}, \tag{18}$$

$$\Lambda_{ij} = \frac{8}{3}S_i S_j - \frac{4}{3}S_k S^k g_{ij}, \tag{19}$$

$$\varepsilon_{\mathrm{fl}(1)} = \rho(c^2 + \Pi(\rho, e)), \quad P_{\mathrm{fl}(1)} = \rho^2\partial\Pi/\partial\rho. \tag{20}$$

Here, the $\varepsilon_{\mathrm{fl}}$ is the perfect fluid density; P_{fl} is its pressure; $\Box$ and ∇_i are D'Alembertian operator and covariant derivative of the Riemannian space, respectively; $\Psi = \kappa(4\pi\alpha_s - \xi\kappa\Phi^2)^{-1}$; $\Theta = k\rho$; $V' = \partial V/\partial\Phi$. By contracting Eq. (15) with u^i and using Eqs. (8), (10) – (12), we find

$$2k_1 = -(\varepsilon_{\mathrm{fl}(1)} + P_{\mathrm{fl}(1)}). \tag{21}$$

Finally, Eqs. (8) and (9) give

$$(\Theta u^i)_{;i} = -(\varepsilon_{\mathrm{fl}(1)} + P_{\mathrm{fl}(1)}). \tag{22}$$

Excluding torsion with the help of Eq. (6), a closed subsystem of Eqs. (5), (7) and (22) is derived. In the framework of GR, this subsystem describes the gravitational interactions of two perfect fluids and a non-minimally coupled scalar field with the potential $V(\Phi)$.

2. Exact Cosmological Solutions

For homogeneous isotropic closed models with the metric

$$ds^2 = a^2(\eta)[d\eta^2 - dr^2 - \sin^2 r(d\theta^2 + \sin^2\theta\, d\varphi^2)] \tag{23}$$

Eqs. (5) and (7) take the form

$$2\frac{a''}{a} - \frac{a'^2}{a^2} + 1 = \Psi[2\xi\Phi\Phi'' + 2\xi\frac{a'}{a}\Phi\Phi' +$$

$$+ \Big(-\frac{1}{2} + 2\xi + 3\xi^2\Phi^2\Psi)\Phi'^2 + \alpha_s a^2 V(\Phi) -$$

$$- 12 (\pi \Theta a)^2 \Psi] - 4 \pi \alpha_s \Psi a^2 (P_{fl(1)} + P_{fl(2)}), \tag{24}$$

$$\frac{a'^2}{a^2} + 1 = \Psi\left[\left(\frac{1}{6} - \xi^2 \Phi^2 \Psi\right)\Phi'^2 + 2\xi \frac{a'}{a} \Phi \Phi' + \right.$$

$$\left. + \frac{1}{3} \alpha_s a^2 V(\Phi) + 4 (\pi \Theta a)^2 \Psi\right] +$$

$$+ \frac{4}{3} \pi \alpha_s \Psi a^2 (\varepsilon_{fl(1)} + \varepsilon_{fl(2)}), \tag{25}$$

$$(1 - 6 \xi^2\Phi^2\Psi) (\Phi'' + 2 \frac{a'}{a} \Phi') + \alpha_s a^2 V' -$$

$$- 12 \pi \alpha_s \xi a^2 \Phi \Psi (\Theta u^k)_{;k} + 6 \xi \left(\frac{a''}{a} + 1\right) \Phi +$$

$$+ 24 \pi \xi \Psi^2\Phi (- \kappa^{-1} \alpha_s \xi \Phi'^2 + \pi a^2\Theta^2) = 0, \tag{26}$$

where the prime denotes the differentiation with respect to η.

We confine ourselves to considering the perfect fluid, which induces torsion with the vacuum equation of state: $P_{fl(1)} = - \varepsilon_{fl(1)}$. Then, from (20) and (22), we obtain

$$\varepsilon_{fl(1)} = C_1, \quad \Theta = C_\Theta a^{-3}, \tag{27}$$

where C_1, C_Θ are integration constants, $(C_1 > 0)$.

For the stiff fluid, we have

$$\varepsilon_{fl(2)} = P_{fl(2)} = C_2 a^{-6}, \qquad C_2 = \text{const}. \tag{28}$$

The scalar field potential $V(\Phi)$ is chosen in the form:

$$V(\Phi) = \beta \frac{\mu^2}{2} \Phi^2 + \frac{\lambda}{4!} \Phi^4. \tag{29}$$

Here, μ and λ are constants; $\beta = + 1$ conforms to a massive scalar field, $\beta = - 1$ corresponds to a Higgs-type nonlinearity.

The exact solutions have been obtained for positive value of the Einstein's effective constant

$$\kappa_{eff} = \frac{\kappa}{1 - \dfrac{\alpha_s \xi}{4 \pi} \kappa \Phi^2} > 0, \tag{30}$$

provided that

$$\lambda = \frac{6}{\pi} \xi^2 \kappa^2 C_1, \qquad \mu^2 = 4 \kappa |\xi| C_1. \tag{31}$$

2.1. Solutions with a 'Gravitational“ Scalar Field

The exact partial solutions have been obtained under the supplementary condition $\sqrt{3}\, \kappa^2 C_1 C_\Theta = 4$. For $\xi < 0$, $\beta = - 1$, the solution is

$$a = a_0 \sqrt{1 + \lambda_1^2 t^2}, \quad \Phi = B \frac{\lambda_1 t}{\sqrt{1 + \lambda_1^2 t^2}}, \tag{32}$$

were $a_0 = a (t = 0) = (2/ \kappa C_1)^{1/2}$, $\lambda_1 = (\kappa^2 c/ 2) \times$ $\times (C_1^3 C_2 |\xi|)^{1/2}$, $B = (4 \pi/ \kappa |\xi|)^{1/2}$, t is the cosmological time $(a (\eta)\, d\eta = c dt)$.

Solution (32) corresponds to the non-singular model with $a_{min} = a_0$, $\Phi_0 = \Phi (t = 0) = 0$, and the asymptotics: $a|_{t \to \pm \infty} \sim \pm t$, $\Phi|_{t \to \pm \infty} \to \pm B$.

For $\xi > 0$, $\beta = 1$, the solution may be written as follows:

$$a = a_0 \sqrt{1 - \lambda_1^2 t^2}, \quad \Phi = B \frac{\lambda_1 t}{\sqrt{1 - \lambda_1^2 t^2}}. \tag{33}$$

It is easy to see from (33) that $a_{max} = a_0$, $\Phi_0 = 0$, and the model is singular at $t = \pm \lambda_1^{-1}$ ($a = 0$, $\Phi = \infty$).

2.2. Solutions with Material Scalar Field

Exact partial solutions have been obtained under the condition $\kappa^3 C_1^2 (3 \kappa C_\Theta^2 + 8 C_2) = 16$. For $\xi < 0$, $\beta =$ $= + 1$, the solution is

$$a = a_0 \sqrt{1 - \frac{1}{2} \sin^2 \lambda_2 t}, \quad \Phi = B \frac{\mathrm{tg}\, \lambda_2 t}{\sqrt{2 + \mathrm{tg}^2 \lambda_2 t}}, \tag{34}$$

where $\lambda_2 = \sqrt{2}\, \lambda_1$, and the expression for a_0 is the same as above.

Solution (34) describes the oscillating Universe with the period $T = \pi \lambda_2^{-1}$. The scale factor a changes from $a_{min} = a_0/\sqrt{2}$ ($\Phi_{min} = B$) to $a_{max} = a_0$ ($\Phi_0 = 0$).

For $\xi > 0$, $\beta = - 1$, the solution may be written as follows:

$$a = a_0 \sqrt{1 + \frac{1}{2} \sinh^2 \lambda_2 t}, \quad \Phi = B \frac{\sinh \lambda_2 t}{\sqrt{2 + \sinh^2 \lambda_2 t}}. \tag{35}$$

It follows from (35) that this solution conforms to the non-singular model ($a_{min} = a_0$, $\Phi_0 = 0$) and has the asymptotics: $a|_{t \to \pm \infty} \sim e^{\pm \lambda_2 t}$, $\Phi|_{t \to \pm \infty} \sim \to \pm B$.

3. Solutions for Two-Fluid Cosmological Models without Scalar Field

Now let us consider the closed models with two-fluid without scalar field. The general exact solution of the Einstein – Cartan's field equations is obtained by quadrature:

$$\int \frac{a^2 da}{\sqrt{s_1 a^6 - a^4 + s_2 + s}} = \pm\, ct, \tag{36}$$

where $s_1 = \frac{1}{3}\kappa C_1$, $s_2 = \frac{1}{3}\kappa C_2$, $s = \frac{1}{4}\kappa^2 C_\Theta^2$.

The analysis of solution (36) shows that, for any s, s_1, s_2, besides the case where these parameters are connected with the constant D by relation

$$s_1 D^3 - D^2 + s_2 + s = 0, \tag{37}$$

the solution has the asymptotics

$$a\,|_{t\to 0} \sim t^{1/3}, \qquad a\,|_{t\to\pm\infty} \sim e^{\pm\,\sigma t}, \tag{38}$$

where $\sigma = c s_1^{1/2}$.

Under condition (37), cosmological models of two types are possible:

1) a non-singular model with $a_{\min} = D^{1/2}$ under the additional condition $0 < D < 2\,(3\,s_1)^{-1}$ and the exponential asymptotic of type (38);

2) a singular model with the asymptotics

$$a\,|_{t\to 0} \sim t^{1/3}, \qquad a\,|_{t\to\pm\infty} \to D^{1/2}, \tag{39}$$

where $D = 2\,(3\,s_1)^{-1}$, $s + s_2 = 4\,(27\,s_1^2)^{-1}$.

It is necessary to note that, for singular models of type (38) and (39), the solutions have been obtained in two cards: $t \in (-\infty, 0)$ and $t \in (0, +\infty)$.

Conclusion

In this article, a variant of the Einstein – Cartan theory with two sources of torsion, the perfect fluid and non-minimal coupling scalar field, has been constructed. The proposed theory can be treated as a generalization of the standard Einstein – Cartan theory with one torsion source.

As an application of the theory, isotropic closed cosmological models with allowance for the stiff fluid have been considered.

In the one-torsion case where the torsion is generated by a perfect fluid with the vacuum equation of state, the general exact solution in quadratures has been obtained. It is shown that both singular and non-singular cosmological model are possible.

In the two-torsion case, the partial exact solutions have been obtained. It is detected that, as distinct from the one-torsion one, the number of non-singular models increases and an oscillating model is possible, in particular.

In the one-torsion case, the presence of the stiff fluid does not tell decisively on the evolution of models. But, in the two-torsion one, all solutions have been obtained only when the stiff fluid is taken into account.

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