LÈVY MOTION OF A PLANE ROTATOR

A.V. CHECHKIN, V.Yu. GONCHAR

National Scientific Center "Kharkiv Institute of Physics and Technology", Institute for Theoretical Physics
(1, Academichna Str., Kharkiv 61108, Ukraine; e-mail: achechkin@kipt.kharkov.ua)

UDC 533.95

As an example of the Lèvy motion, we consider the relaxation of a plane rotator influenced by a stable random process. By solving the fractional Fokker - Planck kinetic equation with fractional derivative with respect to the angular velocity, we get an expression for relaxation of the polar angle cosine and discuss peculiarities of the relaxation regime. We also perform a numerical simulation based on solution of the stochastic Langevin equations with a stable process and demonstrate a quantitative agreement between analytic and numerical results.

In this paper, we consider the relaxation of a plane rotator subjected to symmetric α-stable processes, that is, the processes, whose increments are independent, stationary, statistically self-affine, and stably distributed in the sense of P. Lèvy [1]. These processes are the natural generalization of the Wiener process from the viewpoint based on limit theorems [2], and, thus, the theory of dynamical systems influenced by α-stable processes is a natural generalization of the theory of Brownian motion. It is sometimes called "Lèvy motion", see the recent discussion in [3]. We have, at least, three motivations for our studies.

1. A plane rotator is, obviously, the simplest system, which admits analytic treatment and allows us to discuss the kinetic description of Lèvy motion with the use of the fractional Fokker–Planck equation containing a fractional velocity derivative. In fact, kinetic equations with fractional velocity derivatives have attracted less attention than kinetic equations with a fractional time or space derivative, see the review [4] and references therein.

2. We believe that the systems influenced by α-stable processes can mimic strong collisions with the surrounding medium, when there exist long angle excursions without any characteristic length. These effects cannot be described within the model of Brownian rotator, since the Wiener process can mimic only weak collisions, which produce small angle excursions. It is known, for example, that the magnetic systems can have strong and very strong collisions [5], which are described within different theoretical frameworks [6]; however, the direct comparison with other approaches to the description of strong collisions for rotary random walks is, of course, not the purpose of our communication.

3. We also note that the Lèvy motion of a plane rotator was considered recently in [7]. The author has developed a functional approach to the description of systems influenced by α-stable processes and considered a plane rotator as an example of this approach. Thus, it is of interest to present here another approach to this problem.

Following the classical method of the Brownian motion theory [8], we can write the Langevin equations for a plane rotator subjected to an α-stable process as

$$d\phi = \Omega dt,\; d\Omega = -\nu\Omega dt + L\left(dt\right), \tag{1}$$

where ϕ, Ω are the angle and angular velocity, respectively, $\{\phi, \Omega\} \in (-\infty, \infty)$, $d\phi$ and $d\Omega$ are their increments during a time interval dt, which is assumed to be shorter than time intervals during which physical parameters change appreciably. Further, ν is the friction coefficient, $L\left(t\right)$ is a symmetric α-stable process with the characteristic function [9]

$$\hat{w}_L\left(k,t\right) = \left\langle e^{ikL}\right\rangle = \exp\left(-D\left|k\right|^{\alpha} t\right), \tag{2}$$

where α is the Levy index, $0 < \alpha \leq 2$, and $D > 0$, $D^{1/\alpha}$ is called the scale parameter. Physically, D serves as the measure of the intensity of an external random source in the Langevin equation. It was shown in [10] that the Langevin description of a dynamical system driven by a symmetric α-stable process is equivalent to the kinetic description with the help of the fractional Fokker – Planck equation containing a fractional velocity derivative. In our case, the equation for the probability density function (PDF) $f\left(\phi, \Omega, t\right)$ is

$$\frac{\partial f}{\partial t} + \Omega\frac{\partial f}{\partial \phi} = \nu\frac{\partial}{\partial \Omega}\left(\Omega f\right) + D\frac{\partial^{\alpha} f}{\partial\left|\Omega\right|^{\alpha}}. \tag{3}$$

We use the symmetric fractional derivative of order $\alpha > 0$, which may be defined "for a sufficiently well-behaved" function $g\left(\Omega\right)$, $\Omega \in R$, as the (pseudo-differential) operator characterized in its Fourier representation by

$$\frac{d^{\alpha}}{d\left|\Omega\right|^{\alpha}} g\left(\Omega\right) \div -\left|k\right|^{\alpha}\hat{g}\left(k\right). \tag{4}$$

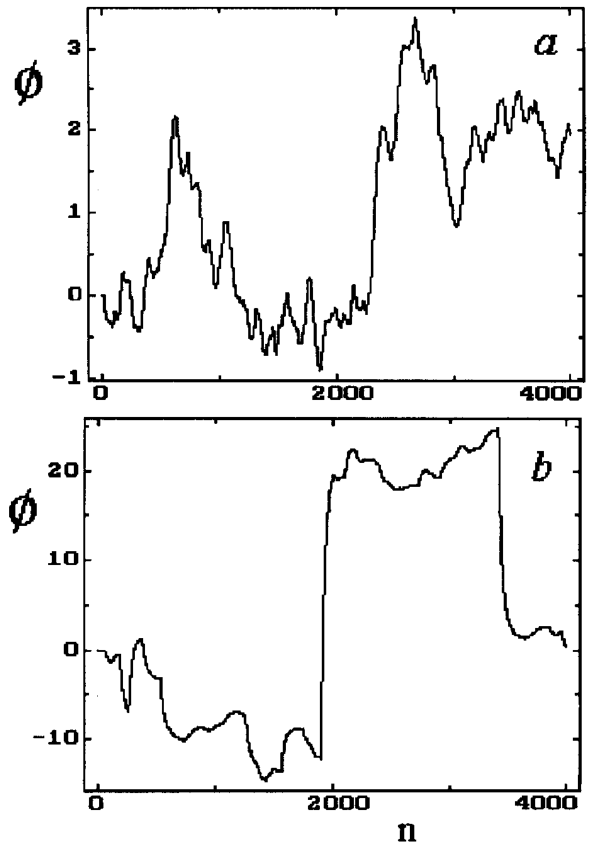

Fig. 1. Polar angle vs discrete time for (*a*) Brownian rotator, and (*b*) Cauchy rotator

Here, on the left-hand side, we adopt the notation introduced in [11]. For the relation between a symmetric fractional derivative and Riemann – Liouville derivatives see [3, 11, 12]. Eq.(3) is solved on the infinite plane $\{\phi, \Omega\}$ with the initial condition

$$f(\phi, \Omega, t = 0) = \delta(\phi - \phi_0)\,\delta(\Omega - \Omega_0). \tag{5}$$

To solve Eqs. (3), (5), we pass to the characteristic function

$$\hat{f}(\kappa, k, t) = \left\langle e^{i\kappa\phi + ik\Omega} \right\rangle, \tag{6}$$

which obeys the equation

$$\frac{\partial \hat{f}}{\partial t} + (\nu k - \kappa)\frac{\partial \hat{f}}{\partial k} = -D\,|k|^{\alpha}\,\hat{f} \tag{7}$$

with the initial condition

$$\hat{f}(\kappa, k, t = 0) = \exp(i\kappa\phi_0 + ik\Omega_0). \tag{8}$$

A solution to Eqs. (7), (8) is obtained by the method of characteristics,

$$\hat{f}(\kappa, k, t) = \exp\{i\kappa\left(\phi_0 + \frac{\Omega_0}{\nu}\right) + i\Omega_0 e^{-\nu t}\left(k - \frac{\kappa}{\nu}\right) -$$

$$-D\int_0^t d\tau \left|\frac{\kappa}{\nu} + \left(k - \frac{\kappa}{\nu}\right)e^{-\nu\tau}\right|^{\alpha}. \tag{9}$$

In case of a Brownian rotator, $\alpha = 2$, from Eq. (9), we got the results presented in [5]. Eq. (9) is in complete analogy with Eq. (39) in [10] describing the force-free relaxation of a Levy particle. Eq. (9) allows us to pass to simpler PDFs, namely,

$$f(\Omega, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ik\Omega} \hat{f}(\kappa = 0, k, t), \tag{10}$$

which describes a relaxation of the angular velocity, as well as to

$$f(\phi, t \mid \phi_0, \Omega_0) = \int d\Omega f(\phi, \Omega, t \mid \phi_0, \Omega_0) =$$

$$= \int \frac{d\kappa}{2\pi} e^{-i\kappa\phi} \hat{f}(\kappa, t \mid \phi_0, \Omega_0), \tag{11}$$

which describes the angle relaxation.

It follows from Eqs. (9), (11) that

$$\hat{f}(\kappa, t \mid \phi_0, \Omega_0) = \left\langle e^{i\kappa\phi} \right\rangle = \hat{f}(\kappa, k = 0, t) =$$

$$= \exp\{i\kappa\phi_0 + i\kappa\frac{\Omega_0}{\nu}\left(1 - e^{-\nu t}\right) -$$

$$-\frac{D\,|\kappa|^{\alpha}}{\nu^{\alpha}} \int_0^t d\tau \left(1 - e^{-\nu\tau}\right)^{\alpha}\}. \tag{12}$$

The large ϕ asymptotics of the PDF $f(\phi, t \mid \phi_0, \Omega_0)$ is determined by the first nonanalytic term in the series expansion of $\widehat{f}(\kappa, t \mid \phi_0, \Omega_0)$ at small κ [13]. Thus, $f(\phi, t \mid \phi_0, \Omega_0)$ behaves as $|\phi|^{-1-\alpha}$ at large ϕ, thus the moments of the order $q \geq \alpha$ diverge. In these cases, one has to use fractional moments of the order $q < \alpha$ for characterizing the anomalous diffusion of the angle. This point is discussed in [10] in detail by the example of a force-free solution to the fractional Einstein – Smoluchowsky equation.

In order to consider the relaxation in more detail, we perform a numerical simulation based on solution of the Langevin equations (1). To model an α-stable process, we use the method which is described in [14] in detail. The results are presented in Figs.1, 2, and

3, see below. Fig. 1 has an illustrative purpose. It shows the angle ϕ versus t for (a) a Brownian rotator, $\alpha = 2$, and (b) a Cauchy rotator, $\alpha = 1$, respectively. Here, $D = 1$, $\nu = 2, n = t/\Delta t$ is the number of time steps (discrete time), and Δt is the length of a single step, $\Delta t = 0.015$. The large jumps ("Levy flights") are clearly seen in the bottom figure; they are due to the power law asymptotics of the PDF and reflect the influence of strong collisions on the rotator from the surrounding medium. On the contrary, large angle excursions are absent in the Brownian case, which mimics weak collisions only.

Below, we focus attention on the relaxation of the polar angle cosine,

$$\gamma(t) = \langle \cos\phi(t) \rangle , \tag{13}$$

which is an important characteristic in case of random rotation and arises, e.g., in the discussion of magnetic relaxation, dielectric polarization, or infrared absorption [5]. This mean is finite for all α. It follows from Eq. (12) that

$$\gamma(t) = \mathrm{Re}\,\left\langle e^{i\kappa\phi}\right\rangle |_{\kappa=1} = \cos\left[\phi_0 + \frac{\Omega_0}{\nu}\left(1 - e^{-\nu t}\right)\right] \times$$

$$\times \exp\left(-\frac{D}{\nu^{\alpha}}\Sigma_{\alpha}(t)\right), \tag{14}$$

where $\Sigma_{\alpha}(t) = \int\limits_0^t d\tau\,(1 - e^{-\nu\tau})^{\alpha}$. This result coincides with the result of [7], where it was obtained by using the functional approach. However, the physical consequences were not considered, so let us discuss them. There are two relaxation regimes, transient and exponential ones, the latter occurs at $t >> 1/\nu$, when $\Sigma_{\alpha}(t) \approx t$. It is of interest to compare the peculiarities of these regimes for different $\alpha, 0 < \alpha \leq 2$. For this purpose, one can set $D = 1$ in Eq.(14); it implies physically that we consider the role of random sources with different Levy indices but with the same intensities. It is clear from Eq. (14) that, during the exponential regime, the relaxation is more rapid for smaller α at $\nu > 1$ ("strong friction case"), whereas it is vice versa during exponential regime at $\nu < 1$ ("weak friction case"). To look at the relaxation process in more detail, γ is shown versus time t for the different values of α and for the cases of strong and weak friction, see Figs. 2 and 3, respectively. In Figs. 2,*a* and 3,*a*, the linear scale is used, whereas the semilogarithmic scale is employed in Figs. 2,*b* and 3,*b*, the solid curves *1, 2, 3* and *4* correspond to the Levy indices 0.5, 1.0, 1.5 and 2, respectively.

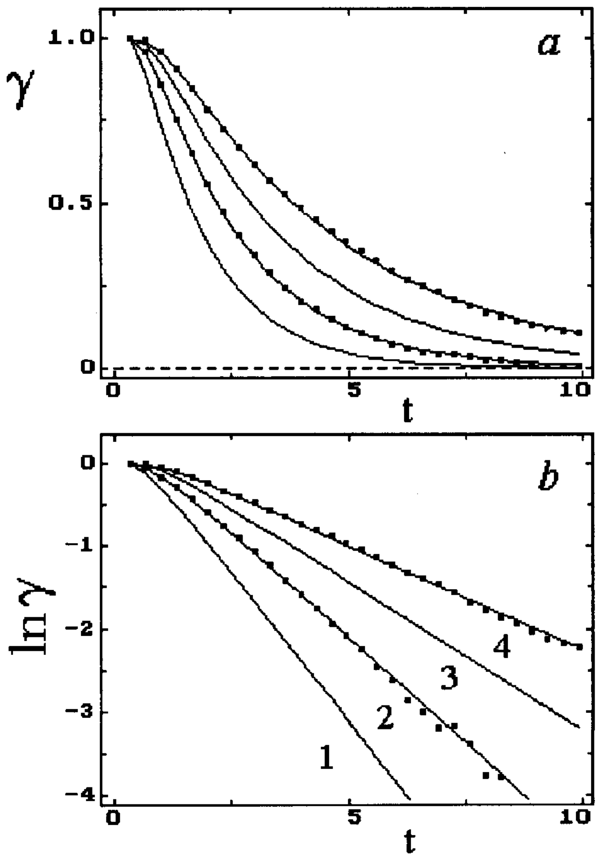

Fig. 2. Mean of the polar angle cosine vs time for the case of strong friction; (*a*) linear scale, (*b*) semilogarithmic scale. The analytic curves *1, 2, 3* and *4* correspond to the Levy indices 0.5, 1.0, 1.5, and 2, respectively, $D = 1, \nu = 2$. The black points indicate the results of the numerical simulation based on solution of the Langevin equations. The statistical averaging is over 20,000 sample paths with the time step $\Delta t = 0.15$

These curves are calculated according to Eq. (14) with $\phi_0 = \Omega_0 = 0$, $D = 1$, $\nu = 2$ and $\nu = 0.22$ for Figs. 2 and 3, respectively. For the strong friction case, the transient regime is short, and the main part of the relaxation process (that is, the time lag, during which γ decreases essentially) has a simple exponential character, which is shown by the segments of straight lines in semilogarithmic scale, see Fig. 2,*b*. With α increasing, the relaxation becomes slower; the most slow relaxation is for the Brownian case, $\alpha = 2$. This effect becomes more and more apparent with ν increasing. Now we turn to Fig. 3. In case of a weak friction, the main part of the relaxation process is within the transient regime, where the difference between relaxation rates is not so clearly seen as in case of large friction. However, in the transient

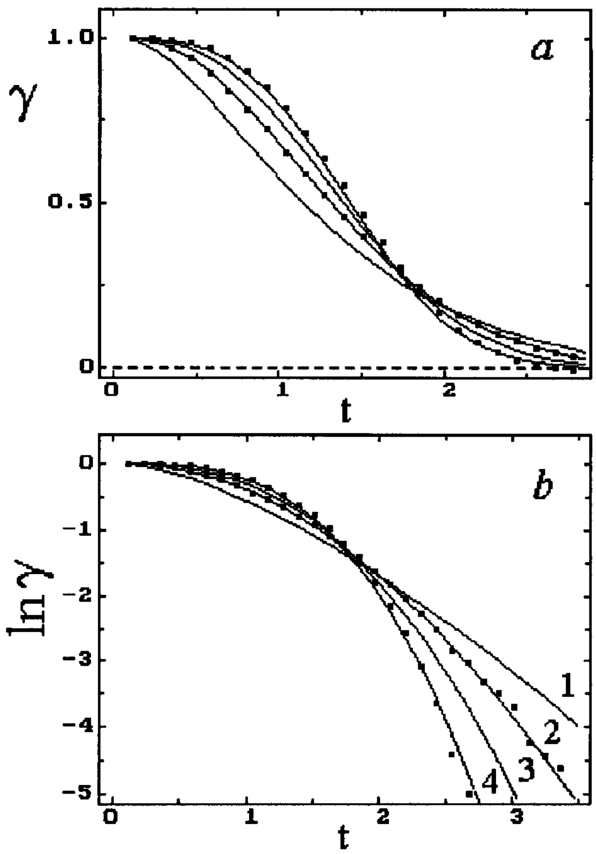

Fig. 3. Mean of the polar angle cosine vs time for the case of weak friction; (*a*) linear scale, (*b*) semilogarithmic scale. The analytic curves *1, 2, 3* and *4* correspond to the Levy indices 0.5, 1.0, 1.5, and 2, respectively, $D = 1, \nu = 0.22$. The black points indicate the results of the numerical simulation based on solution of the Langevin equations. The statistical averaging is over 20,000 sample paths with the time step $\Delta t = 0.001$

regime, the relaxation is more rapid for smaller α, just as in case of strong friction. The exponential regime appears when γ already decreases essentially, and the exponential relaxation becomes more and more slower with α decreasing.

Black points in Figs. 2 and 3 indicate the results of the numerical simulation based on solution of the Langevin equations. In each case, the statistical averaging is over 20,000 sample paths, with the time steps $\Delta t = 0.15$ in Fig. 2 and $\Delta t = 0.001$ in Fig. 3. It is worth to note that the numerical results obtained by statistical averaging over sample paths should not depend on the value of Δt. In the figures, the results of the numerical simulation are shown for two cases $\alpha = 1$ and $\alpha = 2$ only. The results for $\alpha = 0.5$ and $\alpha = 1.5$ also demonstrate a good qualitative agreement between analytic estimates based on solution of the fractional kinetic equation (3) and numerical results based on a numerical solution of the Langevin equations (1).

In summary, we have studied the random Lèvy motion of a plane rotator subjected to an α-stable process. We speculate that this model is interesting not only from the point of view of general statistical physics of non-equilibrium systems, but also can mimic strong collisions with the surrounding medium, which cannot be described within the framework of Brownian motion. The main results are as follows.

1. The fractional Fokker – Planck equation for a plane rotator is solved, and the mean of the polar angle cosine is calculated.

2. The numerical simulation is performed, which is based on a numerical solution of the Langevin equations containing a random source which is an α-stable process. The statistical mean of the polar angle cosine estimated by averaging over numerically generated sample paths is close to the mean obtained analytically. This circumstance confirms the equivalence of the Langevin and kinetic approaches for Levy motion and also evidences in favour of correctness of the used numerical modelling.

3. For the equal intensities of random sources with different Levy indices, the relaxation is more rapid for smaller indices. It implies that long-angle excursions lead to a more rapid "intermixing" of values of the polar angle over the whole axis. This circumstance clarifies the role of strong collisions, which cannot be modelled within the Brownian motion theory.

This work is supported by the Grant INTAS 2000 No.847. Information support within the Project INTAS 98-01 is also acknowledged.

1. *Lèvy P.*//C. R. Acad. Sci. (Paris) – 1936. – **202**. - P.374.
2. *Gnedenko B. V., Kolmogorov A. N.* Limit Distributions for Sums of Independent Random Variables. – Moskva: Izd-vo Tekhniko-Teor. Lit-ry, 1949; Engl. transl.: Massachusets: Addison Wesley Reading, 1954).
3. *Chechkin A. V., Gonchar V. Yu.*// Zhurn. Eksp. Teor. Fiz. – 2000. – **118**, N. 3(9). – P. 730 – 748.
4. *Metzler R., Klafter J.*// Phys. Repts. – 2000. – **339.** – P. 1.
5. *Wyllie G.*//Ibid. - 1980. – **61**, N6. – P. 327 – 376.
6. *Gordon R. G.*//J. Chem. Phys. – 1966. – **44**. - P.1830-1836; *Gordon R. G.*//Adv. Magn. Res. – 1968. – **3**. – P.1 – 42.
7. *Caceres M. O.*// J. Phys. A: Math. Gen. – 1999. – **32**. – P.6009 – 6019.
8. *Chandrasekhar S.*//Rev. Mod. Phys. – 1943. – **15**. – P.1 (Russian Transl.: Stochastic Problems in Physics and Astronomy. Moskva: Inostrannaya Literatura, 1947).

9. *Samorodnitsky G., Taqqu M. S.* Stable Non-Gaussian Random Processes. – New York: Chapman and Hall, 1994.
10. *Chechkin A. V.*//Ukr. Journ. Phys. – 2000. – **45**, N12. – P.1491 – 1497.
11. *Saichev A. I., Zaslavsky G. M.*//Chaos. – 1997. – **7.** – P.753.
12. *Gorenflo R., De Fabritiis G., Mainardi F.*//Physica A. – 1999. – **269.** – P.79.
13. *Zolotarev V. M.* One-Dimensional Stable Distributions. - Moskva: Nauka, 1983 (Engl. transl.: Providence: American Math. Soc., 1986).
14. *Chechkin A. V., Gonchar V. Yu.*//Physica A. – 2000. – **277**, N3 – 4. – P. 312 – 326.

Received 07.03.01

РУХ ЛЕВІ ПЛОСКОГО РОТАТОРА

О. В. Чечкін, В. Ю. Гончар

Р е з ю м е

Як приклад руху Леві розглянуто релаксацію плоского ротатора під впливом випадкового стійкого процесу. Шляхом розв'язування дробового кінетичного рівняння Фоккера – Планка із дробовою похідною по кутовій швидкості одержано вираз для середнього значення косинуса полярного кута та обговорено особливості режиму релаксації. Проведено чисельне моделювання, основане на розв'язкові стохастичних рівнянь Ланжевена із стійким процесом, та показано кількісне узгодження між аналітичними та чисельними результатами.

ДВИЖЕНИЕ ЛЕВИ ПЛОСКОГО РОТАТОРА

А.В.Чечкин, В.Ю.Гончар

Р е з ю м е

Как пример движения Леви рассмотрена релаксация плоского ротатора под воздействием случайного устойчивого процесса. С помощью решения дробного кинетического уравнения Фоккера – Планка с дробной производной по угловой скорости получено выражение для среднего значения косинуса полярного угла и обсуждены особенности режима релаксации. Проведено численное моделирование, основанное на решении стохастических уравнений Ланжевена с устойчивым процессом, и продемонстрировано количественное согласие между аналитическими и численными результатами.