INFLUENCE OF CONSTANT MAGNETIC FIELD ON ENERGY SPECTRUM OF RYDBERG ATOMS

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We develop the hypothesis about the coherent quantum origin of the decameter radiation of the system Jupiter – Io that may explain its very high brightness temperature. The typical magnitude of a magnetic field in the system produces the necessary value of the cyclotron frequency ω_c that is close to the experimentally observed one and corresponds to the quantum transitions between Landau levels. A possible source of this radiation is the transitions in the spectrum of highly excited hydrogen atoms (Rydberg atoms) that has a narrow band of almost equidistant levels (Landau quasi-levels) in a constant magnetic field. This part of the spectrum is calculated by a new nonperturbative method. We discuss also an additional formation mechanism of Rydberg atoms in a dusty plasma.

Introduction

The extremely high pick power in a pulse (about 10^{11} W) of the decameter electromagnetic radiation of the system Jupiter – Io that corresponds to the bright temperature up to 10^{17} K witnesses in favor of the collective coherent regime of generation. The quantum nature of this radiation apparently was first discussed in [1] and was associated with the transitions of electrons between the Landau levels formed in the magneto-active space plasma.

It was suggested that such an inverse population of electrons is produced by external sources: solar radiation, space shock waves, and so on. But any actual realization of the magneto-active plasma with the inverse population of Landau levels as an active medium is hampered by the equidistant character of these levels. In [1], it was assumed that a widening and shifting of Landau levels due to fluctuations of the plasma fields may provide the necessary conditions for the coherent amplification of electromagnetic radiation at the cyclotron frequency ω_c. The necessary intensity of coherent radiation may be gained while it travels through large distances of cosmic systems.

In this paper, we consider the influence of a constant magnetic field on the energy spectrum of highly excited hydrogen atoms. It is shown that, near the ionization threshold, where the energies of Coulomb attraction and magnetic field are equal by magnitude, the electron energy spectrum has a narrow band of levels with equal spacing. This is the so-called quasi-Landau regime that is characterized by a limited set of level spacings $q\frac{\hbar\omega_c}{2}$ (q is an odd integer, ω_c is the cyclotron frequency) [2 – 3].

We would like to note that the planar problem of the energy spectrum of an electron in Coulomb and constant magnetic fields allows one to separate the radial and angular motion but reduces to a multistep recurrent relation. The quantization procedure is not transparent in this case. To solve this particular problem, a special method that may be called the method of auxiliary Hamiltonians [4] was developed. It will be used in this paper. Then we pass to the three-dimensional case and, in conclusion, consider the mechanism of formation of the Rydberg atoms in a dusty plasma in the process of electron-ion recombination on the surface of grains.

1. Energy Spectrum of Hydrogen Atom in Magnetic Field (Two-Dimensional Problem)

Here, we use the method of auxiliary Hamiltonians [4] that has already been successfully applied to the anharmonic oscillator and double-well potentials. It consists in estimating the energy levels of a given problem using the exact energy eigenvalues and eigenfunctions of a sequence of auxiliary Hamiltonians that are practically indistinguishable from the Hamiltonian of interest.

Using the Coulomb units and cylindrical coordinates (ρ, ϕ, z), the main task in the planar case is to solve

a radial Schrodinger equation with the Hamiltonian

$$\hat{h} = -\left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi}\right) + \frac{\alpha^2}{4}\xi^2 + \frac{m^2}{\xi^2} - \frac{2}{\xi}. \quad (1)$$

Here, we have set

$$\rho = \nu\xi, \quad \nu = \frac{\hbar^2}{\mu e^2}, \quad \alpha = \frac{\hbar\omega_c}{2R}, \quad \omega_c = \frac{|e|B}{\mu c},$$

$$R = \frac{\mu e^4}{2\hbar^2} \quad \text{and} \quad \varepsilon = \frac{E}{R} - \alpha m. \quad (2)$$

E is the energy eigenvalue, and B is the applied magnetic field in the z-direction. The angular part of the wave function is $e^{im\phi}$ (m – is the magnetic quantum number). Instead of Hamiltonian (1), we introduce a new auxiliary Hamiltonian

$$\bar{h} = h + 2\gamma\lambda\xi^3 + \lambda^2\xi^4. \quad (3)$$

This Hamiltonian is more complex than (1) but it may be solved exactly by using polynomial solutions. The yet unspecified couplings γ and λ will be found to be positive, extremely small, and quantized. The higher the power of a polynomial the closer the auxiliary solutions to the exact eigenvalues of the initial Hamiltonian (1). The eigenvalues $\bar{\varepsilon}$ of Hamiltonian (3) will thus be stringent upper bounds to those of h. The radial wave functions $\bar{R}$ of (3) will turn out to be polynomials with suitable weights. This will enable us to convert $\bar{\varepsilon}$ into the even further improved estimates of true eigenvalues ε.

Setting

$$\bar{R} = \xi^{|m|}\exp\left[-\left(\beta\xi + \frac{\gamma\xi^2}{2} + \frac{\gamma\xi^3}{3}\right)\right]\sum_{n=0}^{\infty} a_n\xi^n \quad (4)$$

and selecting

$$\gamma^2 + 2\beta\lambda = \frac{\alpha^2}{4}, \quad (4a)$$

we get the folowing 4-term recursion relation with successive coefficients:

$$n(n+p-1)a_n + [2-\beta(p+2n-2)]a_{n-1} +$$

$$+ [\bar{\varepsilon} + \beta^2 - \gamma(p+2n-3)]a_{n-2} + [2\beta\gamma - \lambda(p+$$

$$+ 2n-4)]a_{n-3} = 0, \quad p = 2|m| + 1. \quad (5)$$

Unfortunately, there are no general methods to solve such equations. We note that (5) allows $V = \sum a_n\xi^n$ to be a polynomial of degree k provided that

$$a_k \neq 0, \; a_{k+1} = a_{k+2} = a_{k+3} = 0. \quad (6)$$

Feeding (6) into (5), we immediately get

$$\lambda = 2\beta\gamma/(p+2k+2). \quad (7)$$

Then the conditions $a_{k+2} = 0$ and $a_{k+3} = 0$ be rewritten respectively as

$$[\bar{\varepsilon} + \beta^2 - \gamma(p+2k+1)]a_k + 2\lambda a_{k-1} = 0 \quad (8a)$$

and

$$[2-\beta(p+2k)]a_k + [\bar{\varepsilon} + \beta^2 - \gamma(p+2k-1)]a_{k-1} +$$

$$+ 4\lambda a_{k-2} = 0. \quad (8b)$$

Combining (4) and (7), we get a quadratic equation for γ, whose admissible solution is

$$\gamma = \sqrt{\frac{4\beta^4}{(p+2k+2)^2} + \frac{\alpha^2}{4}} - \frac{2\beta^2}{(p+2k+2)}. \quad (9)$$

Four unknowns β, γ, λ, and $\bar{\varepsilon}$ are uniquely fixed by (7) – (9) such that V is a polynomial of degree k and the wavefunction is characterized by a given value of m (i.e., p). Such solutions, therefore, involve the tuning of the auxiliary couplings γ and λ but the magnetic field α is left arbitrary. Eliminating γ and λ in favor of β by using (7) and (9), the remaining task is to solve (8a) and (8b) for $\bar{\varepsilon}$ and β. Note that all a_k, a_{k-1}, and a_{k-2} can be expressed in terms of $\bar{\varepsilon}$, β, α, p, and k and, through a repeated use of the recursion relation (5), a_0 can be normalized away. We thus arrive at two coupled algebraic equations for $\bar{\varepsilon}$ and β which can be solved for any choice of p, k, and α numerically.

The polynomial solutions of the auxiliary problem and the associated energy eigenvalues can thus be determined. Other solutions of the auxiliary problem are power series solutions that do not interest us here. Our studies will be confined to the case of magnetic fields, for which $\alpha \leq 5\cdot 10^{-5}$.

The outcome of the analysis of (8a) and (8b) can be described as follows.

For k = 0, these equations can be solved analytically, remembering that $a_2 = a_3 = 0$. We have $\beta = 2/p$ and $\bar{\varepsilon} = -\beta^2 + \gamma(p+1)$. Notice that β has exactly the same form as in the pure Coulomb case.

For low lying levels ($n = k + |m|$ are small), the numerical solutions reveal that it suffices to retain only the first terms in (8a) and (8b), so that β remains essentially Coulomb. We have

$$\beta \approx \beta_c = \frac{1}{n+1/2}, \quad \bar{\varepsilon} \approx -\beta_c^2 + 2\gamma(n+1). \quad (10)$$

By expanding γ in powers of α (see (9)), we find

$$\bar{\varepsilon} \approx -\beta_c^2 + \frac{\alpha^2}{4}[20(n+1/2)^2 - 12m^2 + 7] + \ldots \quad (11)$$

in complete accord with the first-order perturbation theory near the Coulomb limit, i.e., for $\alpha \to 0$. The auxiliary couplings being $O(\alpha)$, this result is to be expected.

As we move along the spectrum, n increases, and the expansion of γ in powers of α ultimately breaks down. Here, we have a vivid demonstration of how the Coulomb perturbation theory is naturally vitiated as we move the spectrum towards the threshold, however small α may be. The breakdown occurs below the threshold. This can be checked using (9).

In the neighborhood of the threshold, one finds that β tends to deviate from β_c. However, this departure is rather small. Hence, right up to the threshold, it continues to remain realistic to set β_c. The parameter λ remains exceedingly small. Whereas it was $O(\alpha)$ for low lying levels, it begins to approach the oscillator-dictated value, $\alpha/2$, for higher levels. This is not unexpected. Hence, even at the threshold, the auxiliary couplings $2\gamma\lambda$ and λ are negligible. In essence, the problem of $\bar{h}$ is hardly distinguishable from that of h right up to the threshold region. The energy eigenvalues $\bar{\varepsilon}$ of $\bar{h}$, being strict upper bounds of the corresponding ε of h, are, at the same time, useful estimates of ε. Further, we shall refine on these estimates.

First, with the discussion of the previous paragraph in view, (10) continues to provide an accurate account of the spectrum in the threshold region. With $n \gg 1$, we can write the corresponding relations as

$$\beta \approx 1/n, \quad \varepsilon \approx \bar{\varepsilon} \approx -\beta^2 + 2\gamma n.$$

Hence,

$$\frac{E}{R} \approx -\frac{1}{n^2} + 2\gamma n + \alpha m. \quad (12)$$

For $n \gg |m|$ and $E \approx 0$, e.g., we have

$$2\gamma n^3 \approx 1. \quad (13)$$

But, from (9), on the other hand,

$$\gamma \approx \sqrt{\frac{1}{n^6} + \frac{\alpha^2}{4}} - \frac{1}{n^3}. \quad (14)$$

Thus, at the threshold, one finds

$$\alpha n^3 \approx \sqrt{5}. \quad (15)$$

Using (12) – (15), it easily follows that

$$\left.\frac{\Delta E}{\hbar\omega_c}\right|_{E=0} \approx \frac{\sqrt{5}\,\Delta n + \Delta m}{2}. \quad (16)$$

So that, for $\Delta n = \Delta m = 1$,

$$\left.\frac{\Delta E}{\hbar\omega_c}\right|_{E=0} \approx 1.6. \quad (17)$$

Thus, we obtain the level spacing that is very close to 3/2. The case n, $|m| \gg 1$ can be similarly discussed. Several other characteristic spacings can also be read from (16). A more careful numerical analysis of (10) is easily carried out. It reveals that the spacing of $1.5\hbar\omega_c$ is approached in a small energy band $5\hbar\omega_c$ in width immediately above $E = 0$.

This auxiliary contribution to the energy can be efficiently neutralized in a very elementary manner. The auxiliary couplings are very small, and we have the exact associated polynomial eigenfunctions of $\bar{h}$. These functions are clearly good substitutes for the unknown planar eigenfunctions. We can use them to more exactly compute ε. In the spirit of the perturbation theory, one may subtract the leading contribution of auxiliary terms from $\bar{\varepsilon}$, i.e., the quantum mechanical average $\langle 2\gamma\lambda\xi^3 + \lambda^2\xi^4 \rangle$. One then finds that, indeed, the region of the $1.5\hbar\omega_c$ spacing is around $E = 0$. An illustrative sample of energies in the threshold region, so calculated, is displayed in Table for two typical values of α.

We have used (10) in full, and the required integrals have been carried out numerically. A closer look at this limited set of date already reveals the spacing of $1/2\hbar\omega_c$ ($n = 0$, $m = 1$) and $5/2\hbar\omega_c$ ($n = 2$, $m = 1$) in the $E = 0$ region.

The principal features of our studies are best summed up so far before we turn to the $3D$ problem. We have found that the first term in 8a provides an accurate expression for energy that works at least up to the threshold region for the case of realizable field. The behavior of the parameter γ embodied in (9) divides this part of the spectrum into the perturbative and nonperturbative domains. The auxiliary terms make the contribution to be quite small as compared to the oscillator contribution that turns out to be about n ($n \gg 1$) and has been included exactly. This makes

n	m	E	ΔE	n	m	E	ΔE
43	13	3.612	–	34	10	2.705	–
42	12	2.127	1.485	33	9	1.216	1.49
41	11	0.617	1.51	32	8	~0.3055	1.52
40	10	~0.921	1.54	31	7	~1.865	1.56

the subtraction procedure adopted above quite reasonable. With the auxiliary contributions so neutralized, the numerical analysis (see Table) suggests that the guidance formula of Eq. (16) is better replaced by the relation

$$\left.\frac{\Delta E}{\hbar\omega_c}\right|_{E=0} \approx \frac{2\Delta n + \Delta m}{2}. \qquad (18)$$

Although, all spacings of the type $q/2\hbar\omega_c$ (q is an odd integer) are realizable near $E = 0$ via dipole transitions, the density of level pairs that contribute to the $q = 3$ transitions far exceeds those for which $q > 3$. However, on the basis of the population density of a level pair alone, the $q = 1$ transitions should be intense. Next, note that the $q/2\hbar\omega_c$ spacing has emerged as a feature of this separable problem itself. Further, one finds that the spacings are slightly spread out around the values $q/2\hbar\omega_c$, larger for $E < 0$ and smaller for $E > 0$. Hence, it is quite conceivable that the crossover occurs exactly at $E = 0$. Lastly, the spacings also show a small field dependence.

Three-Dimensional Model of a Rydberg Atom in the Magnetic Field

We now turn to the nonseparable 3D case. Here, we have to solve the eigenvalue problem of the operator

$$h \equiv h_0 - \frac{\alpha^2}{4} r^2 \cos^2\theta, \qquad (19)$$

where

$$h_0 = -\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) + \frac{l(l+1)}{r^2} + \frac{\alpha^2}{4} r^2 - \frac{1}{2}. \qquad (20)$$

Spherical polar coordinateds (r, θ, ϕ) have been employed and the dimensionless radial variable in the Coulomb units has been denoted by r.

The eigenvalue problem $h_0 R_0 = \varepsilon_0 R_0$ is amenable to a treatment exactly in parallel to that of the planar case with only changes $p = 2l + 2$, $n = k + l + 1$ now. Thus, instead of (10), we have

$$\beta \approx \beta_c = \frac{1}{n+1}, \quad \varepsilon_0 \approx -\beta^2 + 2\gamma(n + 3/2). \qquad (21)$$

Hence, all the features, discussed for the planar case, re-emerge for the problem of h_0. The problem of h retains these features without any noteworthy modifications. This assertion will now be justified. The difference between h and h_0 is a term proportional to $\alpha^2 r^2 \cos^2\theta$. This renders the problem to be nonseparable and, in principle, the electron can escape. However, α is extremely small for practical fields and the probability of escape should be negligible. Consider, in particular, states dominantly localized in the $\theta \approx \pi/2$ region. For these, $\langle\cos^2\theta\rangle << 1$.

Further, for pure Coulomb states, $\langle r^2\rangle \sim n^4$. The magnetic field can be expected to localize the states even further. Near the threshold, $\alpha n^3 \sim 1$. Combining these facts, we see that $\langle\alpha^2 r^2 \cos^2\theta\rangle << n$. Hence, as compared to the oscillator contribution of r^2 to the energy, this contribution can simply be ignored. Eventually, the quasi-Landau spacings are again given by (18) with $n = k + l + m$.

Conclusions

Our findings can be summed up as follows: the spectrum up to the threshold regime can be reasonably well understood by dividing the full Hamiltonian into a spherically symmetric part and a residual term that breaks spherical symmetry. The Zeeman term can be trivially incorporated. For practical fields, the spherical part contains all the bulk features of the spectrum and can be studied by the conventional quantum approach. The nonspherical part is incapable to change the results so obtained concerning the spacing near the threshold region in any significant manner.

This remains true provided that the magnetic field does not deform low lying wave functions, and the ionization threshold is associated with high principal quantum numbers. In this situation, we deal with Rydberg atoms that are basically planar.

At the end, we would like to indicate that Rydberg atoms may be produced in the process of three-particle recombination. Their lifetime is comparatively large, and the population may be maintained by the Sun's ultraviolet radiation. There is one more mechanism of formation of a non-equilibrium population of excited atoms in the process of electron-ion recombination on the surface of dusty inclusions in a plasma. Dusty particles are available in a space plasma of the system Jupiter – Io due to the volcano activity. It is known that dusty particles may accumulate a negative charge corresponding to 10^2 – 10^3 electrons per grain with a typical size of order of 1 μ. According to [5], the system 'surface – electron – ion" may be treated as a quasimolecule. It may decay into a neutral atom and a charged grain with one electron less. The most probable formation of atoms with the electron energy equal to the energy of an electron on the surface of the grain. If we assume that the coupling energy of an electron on the grain surface is of order of 10^{-2} eV, then hydrogen atoms will be dominantly formed in the state with the principle quantum number $n = 40$ that, according to the results of the numerical calculation given in Table, corresponds to the region of ionization threshold. A dipole transition from this state is possible that corresponds to $\Delta m = 1$. But the

problem of the equal spacing of Landau levels is removed.

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