CONTINUITY EQUATIONS FOR SPIN AND ANGULAR MOMENTUM AND THEIR APPLICATION

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Continuity equations for spin and angular momentum are built. On the basis of the equations, the passage of optical vortices through elliptically deformed fibers is considered. It is also built a refined theory of R.A. Beth’s experiment. The expressions for elemental torque and force acting on an absorbing particle are obtained.

1. Introduction

The interest to an old problem concerning mechanical action of radiation upon matter has recently gained a new impetus when the ability of vortical laser radiation to trap microparticles was established [1, 2]. Since a rigorous theoretical analysis of the problem should involve highly complicated computational techniques, it was agreed to make approximating model assumptions to estimate the effect of radiation on microparticles [2]. The reasonability of such an approach is partly justified by the fact that trapped particles were immersed in liquid in many real experiments and therefore experienced an additional practically incalculable convectional action. Though the method of obtaining the basic expressions for elemental torque was not specified in [2], it has an evident 'microscopic origin“. All such microscopic approaches (see, for example, [3]) have at least two intrinsic difficulties. The first one is connected with a problem of how to determine the motion of an electron in the field of an arbitrary electromagnetic wave, while the second concerns the problem of the final averaging of microscopic equations in order to obtain expressions in terms of phenomenological material constants. Our first aim is to apply macroscopic continuity equations for spin and angular momentum (AM) to the description of light-matter interaction and, on the basis of a general theory, to determine the limits of applicability of the basic equations in [2].

Another aspect of light-matter interaction concerns the influence of matter on the dynamical characteristics of light, in particular, its angular and spin momentum. By its essence, the continuity equation describes as an action of light upon matter, as vice versa. So the very structure of the right side of the continuity equation can provide information on a possible mechanism of generation of light with its own AM. Our another aim is to establish the structure of AM sources and to figure out possible principles of generation of radiation with its own AM.

Our third aim is to apply the obtained continuity equations to the description of the following processes: the passage of light through a $\lambda/4$ plate (the classical R.A.Beth’s experiment [4]) and the passage of a circular optical vortex (CV) through an elliptically deformed weakly guiding fiber.

2. Continuity Equations for Spin and Angular Momentum

As is known from a general theory, electromagnetic field as a classical vector field has both orbital and spin AM whose sum determines the AM of a system [5]. In a guage-invariant form, the density of the AM can be written as $\mathbf{M} = \mathbf{r} \times \mathbf{D} \times \mathbf{B}$. As was demonstrated in [6], this expression implicity comprises as spin as orbital AM. In the absence of matter, the total AM of the field conserves. This conservation law has the form of a continuity equation and, according to Noether’s theorem, reflects the intrinsic symmetries of Maxwell’s equations [5].

When the presence of matter violates the symmetry of a system, the continuity equation acquires a nonzero r.h.s. that describes the variation of radiation’s AM due to the interaction with matter. Microscopic continuity equations for dynamical variables in the presence of matter are widely distributed in the literature (see, e.g., [7]) while their macroscopic analogs are rather difficult to find. Having failed in finding such an equation, we suggest our method of its derivation without claiming any priority.

2. 1. Continuity Equation for AM

Consider Maxwell's equations and material equations in a medium with tensorial $\hat{\varepsilon}$ and $\hat{\mu}$ – dielectric and magnetic permeabilities, respectively:

$$\mathrm{rot}\,\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \quad (1)$$

$$\mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2)$$

$$\mathrm{div}\,\mathbf{D} = 0, \quad (3)$$

$$\mathrm{div}\,\mathbf{B} = 0, \quad (4)$$

$$\mathbf{D} = \hat{\varepsilon}\,\mathbf{E}, \quad (5)$$

$$\mathbf{B} = \hat{\mu}\,\mathbf{H}, \quad (6)$$

$$\mathbf{J} = \sigma \mathbf{E}. \quad (7)$$

Through evident steps from (1) and (2), we can obtain the following equation:

$$\frac{\partial}{\partial t}(\mathbf{r}\times \mathbf{D}\times \mathbf{B}) + \mathbf{r}\times \mathbf{D}\times \mathrm{rot}\,\mathbf{E} +$$

$$+ \mathbf{r}\times \mathbf{B}\times \mathrm{rot}\,\mathbf{H} + \mathbf{r}\times \mathbf{J}\times \mathbf{B} = 0. \quad (8)$$

To bring (8) to the form of a continuity equation, consider two vector fields $\mathbf{a}$ and $\mathbf{b}$ connected by $\mathbf{b} = \hat{f}\,\mathbf{a}$, where $\hat{f}$ is an $\mathbf{r}$-dependent symmetric tensor. We have:

$$(\mathbf{r}\times \mathbf{b}\times \mathrm{rot}\,\mathbf{a})_i = \varepsilon_{ijk}\left(x_j b_l \frac{\partial a_l}{\partial x_k} - x_j b_l \frac{\partial a_k}{\partial x_l}\right), \quad (9)$$

where ε_{ijk} is the absolutely antisymmetric tensor and all the lower indices vary from 1 to 3. Since $f_{ik} = f_{ki}$, the first term in the r.h.s. of (9) can be brought to the form: $\frac{\partial}{\partial x_k}\left(\frac{1}{2}\varepsilon_{ijk} x_j \mathbf{a}\cdot \mathbf{b}\right) - \frac{1}{2}\varepsilon_{ijk} x_j a_m \times \frac{\partial f_{mn}}{\partial x_k} a_n$. If we suppose that $\mathrm{div}\,\mathbf{b} = 0$, the second term can be written as $-\frac{\partial}{\partial x_l}(\varepsilon_{ijk} x_j a_k b_l) + (\mathbf{b}\times \mathbf{a})_i$. Thus, we have:

$$(\mathbf{r}\times \mathbf{b}\times \mathrm{rot}\,\mathbf{a})_i = \frac{\partial}{\partial x_k}\left(\varepsilon_{ijk}\frac{1}{2} x_j \mathbf{a}\cdot \mathbf{b}\right) -$$

$$- \frac{\partial}{\partial x_l}(\varepsilon_{ijk} x_j a_k b_l) - \frac{1}{2}\varepsilon_{ijk} x_j a_m \frac{\partial f_{mn}}{\partial x_k} a_n + (\mathbf{b}\times \mathbf{a})_i\,. \quad (10)$$

This expression can be written also as

$$(\mathbf{r}\times \mathbf{b}\times \mathrm{rot}\,\mathbf{a})_i = \frac{\partial}{\partial x_k}\left(\varepsilon_{ijl} x_j \left[\frac{1}{2}\mathbf{a}\,\mathbf{b}\,\delta_{kl} - a_l b_k\right]\right) +$$

$$+ (\mathbf{b}\times \mathbf{a})_i - \frac{1}{2}\varepsilon_{ijk} x_j \left(\mathbf{a}, \frac{\partial \hat{f}}{\partial x_k}\mathbf{a}\right), \quad (11)$$

where (...,...) stands for scalar product.

Using (11), we can bring (8) to the form of a continuity equation with right-hand side:

$$\frac{\partial M_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = \frac{1}{2}\varepsilon_{ijk} x_j \left[\left(\mathbf{E}, \frac{\partial \hat{\varepsilon}}{\partial x_k}\mathbf{E}\right) + \left(\mathbf{H}, \frac{\partial \hat{\mu}}{\partial x_k}\mathbf{H}\right)\right] +$$

$$+ (\mathbf{E}\times \mathbf{D})_i + (\mathbf{H}\times \mathbf{B})_i - (\mathbf{r}\times \mathbf{J}\times \mathbf{B})_i\,, \quad (12)$$

where $\Pi_{ik} = \varepsilon_{ijl} x_j T_{lk}$, $T_{lk} = \frac{1}{2}(\mathbf{E}\cdot \mathbf{D} + \mathbf{H}\cdot \mathbf{B})\,\delta_{lk} - E_l D_k - H_l B_k$ is Maxwell's tension tensor. The r.h.s. of (12) represents the density of the AM sources and also the density of the torque induced in matter. The first term describes ponderomotive torque, the second and third terms take account of local nongyrotropy, and the last term describes the torque induced by light pressure. In the case of monochromatic fields in nonmagnetic media, the time-averaged (12) takes the form:

$$\frac{\partial \langle \Pi_{ik}\rangle}{\partial x_k} = \frac{1}{2}\varepsilon_{ijk} x_j \langle(\mathbf{E}, \frac{\partial \hat{\varepsilon}}{\partial x_k}\mathbf{E})\rangle +$$

$$+ \langle \mathbf{E}\times \mathbf{D}\rangle_i - \langle \mathbf{r}\times \mathbf{J}\times \mathbf{B}\rangle_i, \quad (13)$$

where $\langle\grave{}\ \rangle$ means time-averaging. The averaging of bilinear combinations of monochromatic fields is carried out by the rule: $\langle ab\rangle = \frac{1}{2}\mathrm{Re}\,ab^*$.

It is worth noting that is is impossible to divide neither AM density nor AM flux density into pure orbital and spin contributions, though, for the total AM and AM flux, such a separation is possible as in the paraxial approximation [8 – 10] as in the general case [11]. Nevertheless, there is an additional integral of motion for Maxwell's equations that describes spin properties of the field in a local sense.

2.2. Continuity Equation for Spin Momentum

Consider Eqs. (1) – (4) and their complex conjugation. It is possible to obtain the following equation in vacuum:

$$\frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E}^* \times \mathbf{E} + \mu_0 \mathbf{H}^* \times \mathbf{H}) =$$

$$= (\mathbf{E}^* \times \mathrm{rot}\,\mathbf{H} - \mathbf{H}^* \times \mathrm{rot}\,\mathbf{E}) - \mathrm{c.c.} \quad (14)$$

With account of the evident formula $(\mathbf{a} \times \mathrm{rot}\,\mathbf{b})_i = a_j \frac{\partial b_j}{\partial x_i} - \frac{\partial}{\partial x_j}(a_j b_i)$, where $\mathbf{a},\mathbf{b}$ have the property $\mathrm{div}\,\mathbf{a} = \mathrm{div}\,\mathbf{b} = 0$, one can bring (11) to the form of a continuity equation:

$$\frac{\partial S_i}{\partial t} + \frac{\partial \Sigma_{ik}}{\partial x_k} = 0, \quad (15)$$

where $\mathbf{S} = -\frac{i}{2}(\varepsilon_0 \mathbf{E}^* \times \mathbf{E} + \mu_0 \mathbf{H}^* \times \mathbf{H})$, $\Sigma_{ik} = -\frac{i}{2}(\mathbf{E} \cdot \mathbf{H}^* \delta_{ik} + E_k^* H_i - H_k^* E_i) = \mathrm{c.c.}$ We interpret $\mathbf{S}$ as a vector of classical spin density, Σ_{ik} – as a tensor of classical spin flux density. Note that $\mathbf{S}$ has a dimension of energy density and Σ_{ik} – the one of Poynting's vector.

The expression for $\mathbf{S}$ is somewhat similar to the corresponding expressions for spin AM density in [8, 11]. For pure real fields (i.e., without phase shifts),$\mathbf{S}$ also equals zero, so it describes polarization properties of fields as well. But unlike the corresponding vectors in [8, 11], $\mathbf{S}$ has local sense and is an integral of motion with a conservation law in the form of (15).

It can be easily demonstrated that, for a plane elliptically polarized monochromatic wave, the tensor Σ_{ik} has the only nonzero component Σ_{33} (z-axis is aligned with the direction of propagation), and its value equals to the third Stokes parameter s_3 of the same wave [12]. At the same time, Σ_{ik} is an object more rich in content than scalar Stokes parameters. For instance, it is impossible to give any satisfactory description of polarization of a superposition of back and forward waves in terms of Stokes parameters, while we have the standard description in terms of Σ_{ik}. Indeed, for the spin flux density of a plane wave with $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$, $\mathbf{E}_0 = (A_x^{(+)}\mathbf{n}_x + B_y^{(+)}\mathbf{n}_y)\, e^{ikz} + (A_x^{(-)}\mathbf{n}_x + B_y^{(-)}\mathbf{n}_y)\, e^{-ikz}$, we obtain

$$\Sigma_{33} = \Sigma_{33}^{(+)} - \Sigma_{33}^{(-)},$$

$$\Sigma_{33}^{(\pm)} = -\sqrt{\frac{\varepsilon_0}{\mu_0}}\, |A_x^{(\pm)} B_y^{(\pm)}|\, \sin(\arg A_x^{(\pm)} - \arg B_y^{(\pm)}), \quad (16)$$

where A, B are complex constants, $(\pm)$ correspond to forward and back waves, respectively. Let us apply the obtained equations to the description of certain model situations.

3. Locally Isotropic Nonabsorbing Media

Consider the case of monochromatic radiation propagating in an inhomogeneous locally isotropic nonmagnetic medium along the z-axis. A continuity equation assumes the form:

$$\frac{\partial \langle \Pi_{ik} \rangle}{\partial x_k} = \frac{1}{2}\langle E^2 \rangle (\mathbf{r} \times \nabla)_i\, \varepsilon. \quad (17)$$

As evident, the operator $\mathbf{r} \times \nabla$ in (17) is proportional to the quantum mechanical operator $\hat{\mathbf{L}}$ of the orbital AM. When the system possesses the axial symmetry, that is $\varepsilon(\mathbf{r}) = \varepsilon(r,z)$, we obtain $\frac{\partial \langle \Pi_{ik} \rangle}{\partial x_k} = 0$ from (17), since $\hat{L}_3 \sim \frac{\partial}{\partial \varphi}$. From the last equality, the conservation of the z-component of the total flux through a cross-section of the medium follows. Suppose that the field strength rapidly decreases at the infinity from the axis. Then, integrating this equality over an infinite volume limited by two cross-sections transverse to the z-axis, we obtain $K_3 \equiv \iint_S \Pi_{33}\, dS_3 = \mathrm{const}$ with $dS_3 = dxdy$. For paraxial beams in vacuum (in the paraxial approximation, $\Pi_{33} = M_3$), it results in the conservation of $\iint_S M_{33}\, dS_3$ demonstrated for arbitrary fields in [8].

If $\varepsilon = \varepsilon(r, \varphi, z)$, the system has no axial symmetry and K_3 does not conserve. In this case, one can derive an equation describing a variation of K_3. Indeed, integrating (17) with $i = 3$ over a volume limited by two infinitely closely spaced cross-sections, we obtain:

$$\frac{d\langle K_3 \rangle}{dz} = \frac{1}{2}\iint_S \langle E^2 \rangle \frac{\partial \varepsilon}{\partial \varphi}\, dS_3. \quad (18)$$

The r.h.s. in (18) depends on z, so we have a differential equation for the AM flux. Note that, in the basis of circular polarization $|\Psi\rangle \equiv \begin{pmatrix} E_+ \\ E_- \end{pmatrix}$, $E_\pm = \frac{1}{\sqrt{2}}(E_x \mp iE_y)$, Eq. (18) can be written in the paraxial approximation $(E_x, E_y >> E_z)$ as

$$\frac{d\langle K_3 \rangle}{dz} = \langle \Psi |\, \frac{1}{2}\left[\frac{\partial}{\partial \varphi}, \varepsilon\right]_- | \Psi \rangle, \quad (19)$$

where $\langle a | b \rangle = \iint_S (a_1^* b_1 + a_2^* b_2)\, dS_3$ and $[\,`\,,\,`\,]_-$ stands for commutator, which is in agreement with the corresponding result of [8].

4. Elliptically Deformed Optical Fibers

The problem of the AM transmission through optical fibers involves the investigation of the AM flux stability. Consider the propagation of the AM in weakly guiding fibers. In real fibers, there exist fiber sections with small ellipticity that influences the structure of fields [13] and thus the value of the total AM flux. Let us suppose that a circular optical vortex (CV) [14] propagating in a circular fiber encounters a section of the fiber with small elliptical deformation. We introduce the ellipticity of a fiber by the following substitution: $\tilde{\varepsilon}(x,y) = \tilde{\varepsilon}(r) \to \tilde{\varepsilon}[x(1+\delta), y(1-\delta)]$ where $\tilde{\varepsilon}$ is a symmetric permeability distribution in the nondeformed fiber, $\delta << 1$ is the parameter of deformation. When the propagation of light is a paraxial one, we can use (19) to calculate the variation of the total AM flux K_3. Since, in guiding fibers, the strength of fields rapidly decreases as x, y increase, one can decompose $\tilde{\varepsilon}(x + x\delta, y - y\delta)$ in a series in δ:

$$\varepsilon(x,y) \approx \tilde{\varepsilon}(r) + \frac{\partial \tilde{\varepsilon}}{\partial x} x\,\delta - \frac{\partial \tilde{\varepsilon}}{\partial y} y\,\delta. \tag{20}$$

The presence of the elliptically deformed section perturbs the field $\mathbf{E}_0$ of an ideal fiber: $\mathbf{E}_0 \to \mathbf{E}_0 + \mathbf{E}_1\delta$, and, respectively: $|\Psi_0\rangle \to |\Psi_0\rangle + \delta|\Psi_1\rangle$. Since $\dfrac{\partial \tilde{\varepsilon}}{\partial \varphi} = 0$, to calculate a variation in the first order in δ, it is sufficient to use a zero-order approximation field $\mathbf{E}_0$ in (20). With this assumption, we obtain:

$$\frac{dK_3}{dz} \approx \delta \langle \Psi_0 |\, r \sin 2\varphi \frac{d\tilde{\varepsilon}}{dr} \,| \Psi_0 \rangle. \tag{21}$$

For $CV_l^{\pm}$-vortices, $|\Psi_0\rangle \sim \begin{pmatrix} e^{il\varphi} \\ 0 \end{pmatrix}$ approximately or $\begin{pmatrix} 0 \\ e^{il\varphi} \end{pmatrix}$. Therefore, $\dfrac{dK_3}{dz} \approx 0$ that means the conservation of the total AM flux in the first order in deformation and thus the stability of a CV AM flux. In this approximation, we disregard any corrections to propagation constants.

5. Isotropic Absorbing Media

Consider the propagation of light through an absorbing medium. Since there is no much difference between fixed and free electrons on optical frequencies, we have to take into consideration the last term in (12) even in the absence of the latter. For absorbing media, the complex conductivity is $\sigma = \sigma_1 + i\sigma_2$, where σ_1 causes attenuation and energy losses, while σ_2 usually merges with dielectric permeability ε and renormalizes it. Let us suppose that $\varepsilon = \varepsilon_1 - i\varepsilon_2$. According to (12), the averaged induced torque density in the case of monochromatic radiation is $\langle \mathbf{K}_1 \rangle = \frac{1}{2}\,\mathrm{Re}\,\mathbf{E} \times \varepsilon^* \mathbf{E}^*$. If $\mathbf{E} = (\mathbf{E}_1 + i\mathbf{E}_2)\, e^{i\omega t}$ with real $\mathbf{E}_1$ and $\mathbf{E}_2$, then $\langle \mathbf{K}_1 \rangle = \varepsilon_2 \mathbf{E}_1 \times \mathbf{E}_2$. This part of the total torque equals to zero in the absence of absorption and is responsible for the existence of Sadovskii's effect.

In the case of total absorption, $\sigma_2 = 0$ and the last term in (12) yields $-\sigma_1 \mathbf{r} \times \mathbf{E} \times \mathbf{B}$. For nonmagnetic media, the corresponding torque is $\langle \mathbf{K}_2 \rangle = -\mu_0 \sigma_1 \mathbf{r} \times \langle \mathbf{P} \rangle$, where $\mathbf{P} = \mathbf{E} \times \mathbf{H}$ is Poynting's vector. To calculate energy losses, we can use a well-known result of electrodynamics [15]: $\langle N \rangle = \sigma_1 \langle E^2 \rangle$, where N is the density of power losses. Therefore, in the case of total absorption, we have

$$\frac{\langle \mathbf{K} \rangle}{\langle N \rangle} = \frac{\varepsilon_2 \mathbf{E}_1 \times \mathbf{E}_2 - \mu_0 \sigma_1 \mathbf{r} \times \langle \mathbf{P} \rangle}{\sigma_1 \langle E^2 \rangle}. \tag{22}$$

Though we treat ε_2 and σ_1 as independent parameters, $\varepsilon_2 = \sigma_1/\omega$ for monochromatic fields on optical frequencies, where ω is radiation's frequency [15].

Consider the case where monochromatic radiation is incident on an absorbing particle. The r.h.s of (22) determines the local ratio of the induced mechanical torque to absorbed power. So, to obtain a locally induced torque, one has to multiply the r.h.s. of (22) by the absorbed power. In the case of total absorption, this power is $\langle \mathbf{P} \rangle \cdot d\mathbf{S}$ where $d\mathbf{S}$ is a small area element of the absorbing surface. So, for the locally induced surface torque $d\langle \mathbf{K} \rangle$, we obtain

$$d\langle \mathbf{K} \rangle = \left\{ \frac{1}{\omega} \frac{\varepsilon_0 \mathbf{E}_1 \times \mathbf{E}_2}{\langle \varepsilon_0 E^2 \rangle} - \frac{1}{c^2} \frac{\mathbf{r} \times \langle \mathbf{P} \rangle}{\langle \varepsilon_0 E^2 \rangle} \right\} (\langle \mathbf{P} \rangle \cdot d\mathbf{S}), \tag{23}$$

where c is the speed of light.

In paraxial beams $\varepsilon_0 E^2 \approx \mu_0 H^2$, and we come to the corresponding result of [2]. But this is not true in an arbitrary nonparaxial situation. So, (23) gives an improvement of the basic expression suggested in [2]. As in real experiments on trapping of microparticles, the particles are placed in the beam waist [1, 2], where the z-components of fields are comparable with the transverse ones, one can hope to improve the agreement with experimental data by using (23) instead of the basic formula suggested in [2]. Note that the corresponding expression for the induced force is

$$d\langle \mathbf{F} \rangle = -\frac{1}{c^2} \frac{\langle \mathbf{P} \rangle}{\langle \varepsilon_0 E^2 \rangle} (\langle \mathbf{P} \rangle\, d\mathbf{S}). \tag{24}$$

When $\sigma_2 \neq 0$, the term $i\sigma_2 \mathbf{r} \times \mathbf{E} \times \mathbf{B}$ in the r.h.s. of (12) after time-averaging merges with the corresponding term of the tension tensor, and σ_2 renormalizes dielectric permeability ε as usual. Thus, without any loss of generality, we can assume $\sigma_2 = 0$. In the case of partial absorption, the r.h.s of (23), (24) have to be multiplied by the absorption coefficient q.

6. Passage of Light through a Quarter-wave Plate

As an example of the application of continuity equations, we shall consider the passage of light through a quarter-wave plate. As was demonstrated by R.A.Beth in his classical direct experiment [4], the transformation of a left circularly polarized light into a right one is accompanied by the transfer of the AM from light to matter resulting in the torque acting on the plate. Having no intention to say anything derogatory about this convincing experiment and its theoretical explanation, we would like to draw attention to some possible sources of misunderstanding concerning theoretical description of it. In addition, such a model is almost the only model having the exact and simple solution that enables us to demonstrate in detail that spin and AM continuity equations present complementary descriptions of the process.

6.1. Spin and AM Flow Balance for a Quarter-wave Plate

The continuity equation for the AM for nonmagnetic transparent double refracting media for monochromatic fields has the form:

$$\frac{\partial \langle \Pi_{ik} \rangle}{\partial x_k} = \frac{1}{2} \varepsilon_{ijk} x_j \langle (\mathbf{E}, \frac{\partial \hat{\varepsilon}}{\partial x_k} \mathbf{E}) \rangle + \langle \mathbf{E} \times \mathbf{D} \rangle_i. \qquad (25)$$

To analyze an AM balance, one has to integrate (25) over a volume V. Then the integrated r.h.s. of (25) yields the induced torque. Let us chose the volume V as a right prism with a surface Σ, as shown in Fig.1. The integration of (25) over V yields:

$$\oiint_{\Sigma} \langle \Pi_{ik} \rangle dS_k = \int_V \left\{ \frac{1}{2} \varepsilon_{ijk} x_j \langle \mathbf{E}, \frac{\partial \hat{\varepsilon}}{\partial x_k} \mathbf{E} \rangle + \langle \mathbf{E} \times \mathbf{D} \rangle_i \right\} dV. \qquad (26)$$

Since, $\langle \Pi_{33} \rangle = 0$ on both faces of the prism, the AM flows through the side surface of the prism. The value of the AM flux K_3 is given by

$$\langle K_3 \rangle = \int_0^d dz \langle E_1 E_2 \rangle (\varepsilon_1 - \varepsilon_2) S, \qquad (27)$$

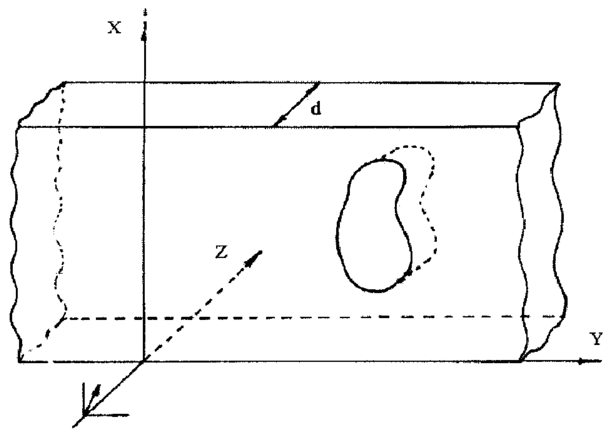

Fig. 1. Orientation of the coordinate system with respect to a quartz plate

where d is the thickness of the plate, $\varepsilon_1, \varepsilon_2$ are the principal values of the dielectric permeability tensor, S is the prism base area. Here, we assume that E_i depends on z only. To calculate the induced torque, one has to find the distribution of $\mathbf{E}$ within the plate.

The problem of light traverse through a birefringent plate is a well-studied problem (see, e.g., [12]). We shall use the solution accounting reflections on both faces of the plate. Suppose that the x-axis of a Cartesian coordinate system is aligned with the principal axis of the plate, and the faces have z-coordinates 0 and d, respectively. The plate divides the space into 3 areas: I ‾ the area before the plate, II ‾ within the plate, III ‾ behind it. The solutions of the Helmgoltz equations for E_1 and E_2 have the form:

$$E_{\mathrm{I}i} = A_i e^{ik_0 z} + C_{1i} e^{-ik_0 z},$$

$$E_{\mathrm{II}i} = C_{2i} e^{ik_i z} + C_{3i} e^{-ik_i z}, \quad E_{\mathrm{III}i} = C_{4i} e^{ik_0 (z-d)}, \qquad (28)$$

where A_i, C_i are constants, $i = 1, 2$; k_1, k_2 are wave numbers for x- and y-polarized waves within the plate, respectively, k_0 is the wave number in the void, coefficients C are as follows:

$$C_{4i} = A_i \delta_i \frac{e^{ik_i d}}{1 - \rho_i e^{2ik_i d}}, \quad C_{3i} = \frac{-2\rho_i A_i}{(1 - n_i)(e^{-2ik_i d} - \rho_i)},$$

$$C_{1i} = A_i \sqrt{\rho_i} \frac{1 - e^{2ik_i d}}{1 - \rho_i e^{2ik_i d}}, \quad C_{2i} = \frac{2A_i}{(1 + n_i)(1 - \rho_i e^{2ik_i d})}, \qquad (29)$$

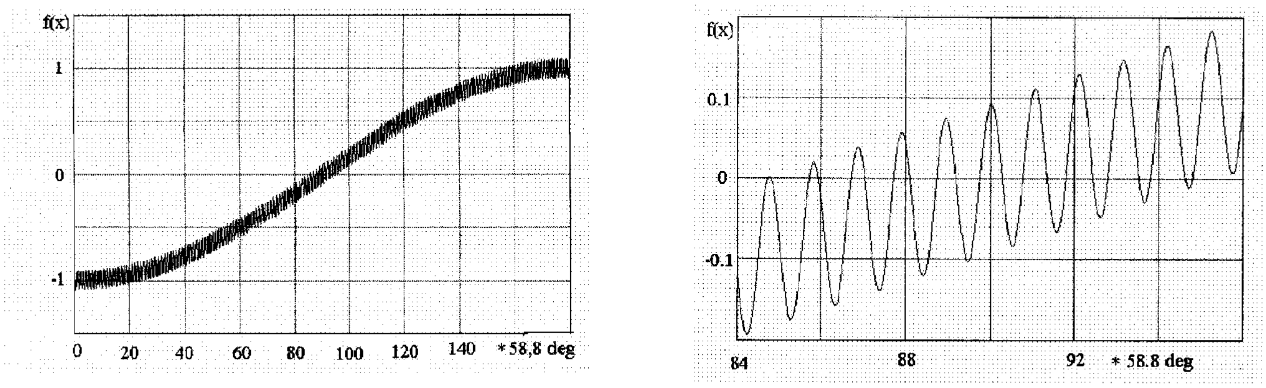

Fig. 2. Graphic solution of Eq. (30b) at $\gamma_1 = 0$ for a quartz plate. Here, $f(x) = 0.093\cos 6.008x - \cos 0.017x$, x in degrees, $d = \frac{x\pi}{180°(n_1 - n_2)\,k_0}$. Fig. 2,*b* depicts the enlarged section of Fig. 2,*a*. The zeros of $f(x)$ correspond to a 90° phase shift, the shift differs from 90° by the order of 1° between neighboring zeros

where $\delta_i = \frac{4n_i}{(1+n_i)^2}$, $\rho_i = \left(\frac{1-n_i}{1+n_i}\right)^2$, n_i is the refractive index for the corresponding axis. Since we can preset the amplitudes of x- and y-polarized waves and the phase shift between them, let us find out the conditions under which an outcoming wave will be circularly polarized. If we assume that $A_1 = A_{01}\,e^{i\gamma_1}$, $A_2 = A_{02}$, where A_{0i} are real amplitudes, such conditions will assume the form:

$$\tilde{A}_{01}\,\delta_1 = \tilde{A}_{02}\,\delta_2, \tag{30a}$$

$$\mathrm{tg}\,(\varphi_1 - \varphi_2) = \mathrm{ctg}\,\gamma_1, \tag{30b}$$

where $\varphi_i = \mathrm{arctg}\left(\frac{1+n_i^2}{2n_i}\,\mathrm{tg}\,k_i d\right)$, $\tilde{A}_{0i} = A_{0i}\,(\delta_i^2 + 4\rho_i \sin^2 k_i d)^{-1/2}$. The first condition provides the equality of amplitudes while the second guarantees a $\pi/2$ phase shift between x- and y-polarized outcoming waves.

To calculate the integral in the r.h.s. of (27), we take into account that, for quartz, $n_1 = 1{,}5533$, $n_2 = 1{,}5442$. We can assume almost everywhere that $n_1 \approx n_2 \equiv n$, $\rho_1 \approx \rho_2 \equiv \rho$, $\delta_1 \approx \delta_2 \equiv \delta$, $k_1 \approx k_2 \equiv k$ and take into consideration that the first set of roots d_i of (30a), (30b) lies in the proximity of $d_0 = \frac{\pi}{2(k_1 - k_2)}$: $d_i = d_0 + \Delta d_i$, $\Delta\, d_i << d_0$. Indeed, for a quartz plate, (30b) leads to a transcendental equation: $\cos\,(k_1 - k_2)\,d = 0{,}093\cos 2kd$. Its graphic solution for $\gamma_1 = 0$ is given in Fig. 2. Evaluating the integral, we disregard terms of order k^{-1}. From (27), (28) through (29), we obtain:

$$\langle K_3\rangle \approx \frac{2\,(\varepsilon_1 - \varepsilon_2)\,\tilde{A}_{01}\,\tilde{A}_{02}\,S}{(k_1 - k_2)(n+1)^2}\,\{\delta\cos\,(\gamma_1 + \varphi_0) -$$
$$- (1+\rho)\sin\,(\gamma_1 + \varphi_0)\}, \tag{31}$$

where $\varphi_0 = \mathrm{arctg}\,\frac{2\rho\sin 2kd}{1-\rho^2}$.

The power N of the outcoming radiation is given by time-averaged Poynting's vector $\mathbf{P}$. With account of (30a), we obtain:

$$\langle N\rangle \approx 4\sqrt{\frac{\varepsilon_0}{\mu_0}}\,S\,\tilde{A}_{01}\,\tilde{A}_{02}\,\frac{\delta}{(n+1)^2}. \tag{32}$$

We also have

$$\frac{\langle K_3\rangle}{\langle N\rangle} = \frac{1}{\omega\delta}\,[\,\delta\cos\,(\gamma_1 + \varphi_0) - (1+\rho)\sin\,(\gamma_1 + \varphi_0)\,]. \tag{33}$$

Let us determine the conditions under which this ratio equals to ω^{-1}. Further, we disregard terms of order ρ^2 (for quartz, $\rho \approx 0.04$). It is possible to demonstrate that, if $\gamma_1 = 0$ (an incident beam is linearly polarized), Eq. (33) has no solutions, i.e., it is impossible to transform a linearly polarized beam into a circularly polarized one with maximum power → torgue transformation coefficient. In other words, a widespread statement that a quarter-wave plate transforms the linear polarization of a photon into a circular one and thus experiences a recoil torque of N/ω is true with an accuracy to ρ.

What really the plate does is the transformation of spin (or spin AM) fluxes. Most explicitly, it can be demonstrated by the following example. If we put $\gamma_1 = -\varphi_0$, $d = d_0$, then (30b) is justified identically, the ratio $\langle K_3\rangle / \langle N\rangle$ equals to ω^{-1}, and (30a) yields the condition for an inclination of the incident beam polarization vector. The analysis of spin fluxes enables us to establish that, at the same time, Σ_{33} in the area before the plate is 0, while it is 1 behind it.

The passage of light through the plate has also transparent quantum mechanical analogies. Indeed, not only the Helmholtz equation for this problem coincides with Schrödinger's equation for over barrier tunneling. Moreover, the continuity of $\mathbf{E}_t$ and $\mathbf{H}_t$ on the faces of the plate is tantamount to the continuity of fields $\mathbf{E}_t$ (wave functions) and their first derivatives on boundaries. Should someone carry out the procedure of secondary quantization of a photon field, he will obtain a similar description of this process in terms of spin currents.

6.2. Passage of a Paraxial Wave Packet through a Quarter-wave Plate

A more realistic model of the process should take account of such factors as incoherence and a finite lateral extent of the beam. For the sake of simplicity, we disregard reflections on the faces. We introduce the dispersion of incident light through the dispersion of the amplitude A_{01} in the k-space: $A_{01} = A\exp\left\{-\dfrac{(k-k_0)^2}{2(\Delta k)^2}\right\}$, where k_0 is the mean wave number of a wave packet, $\Delta k << k_0$ is the standard deviation. Let us assume that the thickness d_0 of the plate is such that the plate operates as a quarter wave plate on the wavelength $\lambda_0 = 2\pi/k_0$. Then we obtain for the total torque with the account of (31) and approximations made:

$$\langle K_3\rangle = \frac{2A^2}{(1+n)^2}\frac{\varepsilon_1-\varepsilon_2}{n_1-n_2}\times$$

$$\times\int_0^{\infty}\exp\left\{-\frac{(k-k_0)^2}{(\Delta k)^2}\right\}\frac{1}{k}\sin\frac{\pi k}{2k_0}\,dk. \quad (34)$$

With the same accuracy, we have:

$$\langle N\rangle = A^2\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_0^{\infty}\left\{-\frac{(k-k_0)^2}{(\Delta k)^2}\right\}dk. \quad (35)$$

Evaluating the integral in (35) by the Laplace method, we obtain

$$\frac{\langle K_3\rangle}{\langle N\rangle} \approx \frac{1}{\omega_0}\exp\left\{-\left(\frac{\pi\,\Delta\lambda}{4\lambda_0}\right)^2\right\}, \quad (36)$$

where $\omega_0 = ck_0$, $\Delta\lambda$ is the standard deviation of the beam with respect to wavelength. For a He‾Ne laser radiation with $\lambda_0 \approx 0.66\,\mu$m, $\Delta\lambda \approx 6\cdot 10^{-3}\,\mu$m, we have the result ω_0^{-1} for the ratio $\langle K_3\rangle / \langle N\rangle$ with an accuracy of 0.5%

To understand the effect of finite lateral extent of a beam on the description of the process, let us consider the traverse of an almost transverse paraxial beam through a plate of finite size. An important characteristic of such beams is a ratio of the total AM flux through a cross-section of the beam to the total power. As was demonstrated in [8], this ratio is

$$\frac{\langle K_3\rangle}{\langle N\rangle} = \frac{1}{\omega}\frac{\langle\Psi\,|\,l_z+\sigma_z\,|\,\Psi\rangle}{\langle\Psi|\,\Psi\rangle}, \quad (37)$$

where $l_z = -i\dfrac{\partial}{\partial\varphi}$, σ_z is the Pauli matrix. The operator l_z describes the total orbital AM of a beam, while σ_z is its total spin AM.

A particular and very important case of such beams is a Laguerre ‾ Gaussian beam. Suppose that a monochromatic linearly polarized LG beam with certain orbital number l is incident on a $\lambda/4$ wave plate that transforms the beam into a circularly polarized one. If also the width of the plate is much greater than the characteristic width of the beam on the plane of its intersection with the plate, one can disregard an AM flux through the side surface of the plate. In this case, the AM flows through the faces of the plate: on the first face, its value is proportional to $\langle\Psi|\,\hat{l}_z\,|\,\Psi\rangle$, while the outcoming beam carries the flux $\sim\langle\Psi|\,l_z+\sigma_z\,|\,\Psi\rangle$. An increase of $\langle\Psi|\,\sigma_z\,|\,\Psi\rangle$ is compensated by the resulting torque of the same value. So the plate changes the spin part of the total AM. It should be noted that, in the paraxial approximation, the total spin flux $\langle\Pi_3\rangle = \iint_S \langle\Sigma_{33}\rangle\,dxdy$ is proportional to $\langle\Psi|\,\sigma_z\,|\,\Psi\rangle$ and

$$\frac{\langle\Pi_3\rangle}{\langle N\rangle} = \frac{\langle\Psi|\,\sigma_z\,|\,\Psi\rangle}{\langle\Psi|\,\Psi\rangle}. \quad (38)$$

Thus, in the case of beams with finite lateral extent, the difference between the descriptions in terms of spin flux and spin AM flux is not so striking as in the ideal case. It is evident that one should also speak about flux transformations rather than density transformations.

If the width of the plate is comparable with the width of the beam, the first term on the r.h.s. of (25) also gives contribution to the induced torque. But if the system has an axial symmetry, this contribution vanishes. In paraxial beams, this term describes a variation in the orbital AM [8] and is zero when

$\partial\hat{\varepsilon}/\partial\varphi = 0$. Otherwise, the resulting torque will have the part due to an orbital AM change. Since there were used round plates in Beth's experiment, the contribution of the first term could arise only due to misalignment.

7. Discussion

At first sight, continuity equations with right hand side have little advantage by comparison with other methods [15], when the question of forces and torque acting on a body in electromagnetic field is considered. Indeed, to integrate the r.h.s. of a continuity equation, one has to know the distribution of fields within the body and this problem is as difficult as the problem of finding such a distribution outside it. Nevertheless, in the case where permeability ε_1 of a homogeneous body slightly differs from medium's permeability ε_2, the r.h.s. of a continuity equation is suitable for the development of perturbation theory with the formal decomposition parameter $\Delta\varepsilon = \varepsilon_1 - \varepsilon_2$. The first term of a perturbation series is obtained when we use zero-approximation (nonperturbed) values of field on the right-hand side. The expressions obtained in such a way for torque and force correspond to a linear response of a field on an external action. For nonabsorbing media, we readily obtain the first-order terms of the decomposition:

$$\langle \mathbf{F}_{(1)} \rangle = \Delta\varepsilon \oiint_S \frac{1}{2} \langle E^2_{(0)} \rangle \, d\mathbf{S};$$

$$\langle \mathbf{K}_{(1)} \rangle = \Delta\varepsilon \oiint_S \frac{1}{2} \langle E^2_{(0)} \rangle \, \mathbf{r} \times d\mathbf{S}, \qquad (39)$$

where S is the bounding surface of the body, subindices specify the order of approximation. Here we have also used the expression for the right side of a continuity equation for the linear momentum: $\frac{1}{2} E^2 \frac{\partial\varepsilon}{\partial x_i}$ (nonabsorbing, locally isotropic media, $\mu = 1$). These expressions may be used in the problem of trapping of biological cells.

What is more evident is that the structure of AM sources provides an insight into basic mechanisms underlying principles of generating the radiation with AM. It also yields a possible classification of such generating devices. According to it, the first group includes the devices in which the generation of AM is linked with energy absorption. Probably the only existing type of such devices is based on the computer synthesized hologram technique [16, 17]. Roughly speaking, a hologram interacts with an incident beam at dark areas where its AM changes in accordance with last three terms in Eq. (12). The specific pattern of a hologram provides approproate boundary conditions for a generating vortex and a selective choice of AM carrying zones of the incident beam as well. Naturally, the hologram experiences a recoil torque in this process.

The second group of such devices generates AM without the intrinsic energy absorption. In this case, the generation is connected with the presence of the first term on the r.h.s. of (12). One can include various mode converters to this class [18 – 20]. As was pointed out in [20], an increase in AM is caused by the ponderomotive interaction between the lens and radiation on a 'Hermite – Gaussian side“ of the lens. This interaction originates due to a difference of the free energy density in the lens and outside it, thus having a 'static“ origin rather than a 'dynamical“ one. The torque acting on a boundary surface element is proportional to $\langle E^2 \rangle \nabla \varepsilon$, where $\langle E^2 \rangle$ describes the energy density of an effective static electric field. An equivalent counteraction brings about radiation's AM. Since this mechanism takes place whenever a sharp boundary is considered, it is theoretically possible to create a system of refracting transparent elements, other than a lens converter, that changes radiation's AM.

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